WAVE PROPAGATION AND ATTENUATION IN AN INFINITE PERIODIC STRUCTURE OF ASYMMETRIC SCATTERERS

Darryl McMahon
Curtin University, Centre for Marine Science and Technology, Bentley, WA 6102, Australia
email: D.Mcmahon@curtin.edu.au

Periodic structures exhibit interesting periodic structure wave (PSW) phenomena although the literature is mostly devoted to symmetric rather than asymmetric systems. Waves in asymmetric structures is a topic of interest for applications where the energy flow needs to be reduced in particular directions. Relatively little has been investigated about wave modes in asymmetric periodic structures that exhibit nonreciprocal wave propagation and attenuation. This paper reports on such an investigation but restricted to an infinite one-dimensional structure of equally spaced nonreciprocal scatterers of structure waves (SW). An infinite structure without boundary effects has wave characteristics the same between every pair (i.e. "cell") of adjacent scatterers, which simplifies the theory to considering only two adjacent cells. It is found that only one type of wave mode, an incoherent energy wave (IEW), can exist in an infinite nonreciprocal periodic structure. In the case of elastic scattering the IEW is a "passing" band in one direction and a "stopping" band in the opposite direction. The IEW is also an allowed mode for symmetric scatterers. However for symmetric scatterers three other modes are also possible for both directions. One is the well-known Bloch-Floquet wave (BFW), which for elastic scattering alternates between passing and stopping bands as a function of wavenumber. The other two "non-BFW" modes result from symmetric moduli of the reflection and transmission scattering coefficients, but asymmetric scattering phase shifts. In the case of elastic scattering one non-BFW is a passing band and the other is a stopping band. In contrast to BFW resulting from multiple reflections and transmissions of a single SW, the IEW and two non-BFW wave modes require two different but correlated SW coupled by scattering.
Keywords: wave, periodic, asymmetric, nonreciprocal, transmission

1. Introduction

Research on nonreciprocal wave transmission, also called unidirectional, asymmetric or wave diode transmission, shows that asymmetric structures can be designed for optical [1,2], electromagnetic and acoustic devices [3,4]. This paper develops theory for PSW that could investigate for instance how a relatively small asymmetry of individual scatterers might lead to a much larger asymmetry for a periodic structure.

This paper uses a scattering approximation for modelling waves in infinite one-dimensional periodic structures of equally-spaced identical scatterers of SW, in contrast to equations of motion for coupled masses or periodic wave equations [5, 6]. Wave transmission, reflection and attenuation ought then be easier to model. Indeed this applies to BFW that are found in the case of symmetric scatterers where the difference in backward reflection and forward transmission phase shifts is $\delta = \pm\pi/2$. BFW are the result of a single SW forward and backward scattered any number of times such that the phase shifts δ do not decohere the overall wave phase. Depending on SW wavenumber the effect of δ either creates the net phase of a travelling wave (passing band) or a standing wave (stopping band). Actually an extension of the scattering model for BFW by taking into account energy absorption weakens the distinction of stopping and passing bands [7, 8].

The same assumptions of the theory for BFW fails when $\delta \neq \pm\pi/2$ because, although it is a simple extension to the characteristic equation (CE), it does not conform to conservation of energy (CoE). A solution to this problem is found by assuming that $\delta \neq \pm\pi/2$ requires two different SW that are coupled by scattering and makes them correlated [8].

It is straightforward to derive a CE for two opposite propagating SW coupled by scattering [8]. For an infinite periodic structure, it is only necessary to consider coupling of SW for two adjacent cells which makes the CE a quadratic equation. The correlation from this coupling is a phase ψ derived from solutions of the CE constrained to satisfy CoE.

It is instructive to distinguish the subject of this paper, which is wave properties for “thick” periodic structures, in contrast to the properties of “thin” periodic structures. The reflectivity of a thin periodic structure may sometimes be approximated by just one backward scattering any distance into the structure. Coherent interference from reflections from different distances leads to wavelength dependent peaks and troughs. This is the case for 1D X-ray crystallography where the strongest reflection peak is from constructive interference for wavelengths equal to twice the scatterer spacing d (i.e. Bragg condition). The forward and backward scattering phase shifts make no difference in this case because the two interfering waves both have one identical reflection, except separated by one scatterer spacing, and equal numbers of forward scatterings in opposite directions for which the phase shifts add to zero.

SW reflections are more complicated for a thick periodic structure because multiple scatterings dominate. In this paper we assume the reflectivity of a finite but thick structure can omit boundary effects and use an infinite periodic structure approximation. Except for BFW, the scattering phase shifts make a large difference through two correlated SW within the structure that smear out Bragg-like reflectivity peaks. SW wavenumber dependence still exists but only because scattering coefficients generally depend on wavenumber which is left implicit in this paper.

Solutions in previous papers limited asymmetry to the scattering phase shifts but still treated the modulus of reflection and transmission coefficients as symmetric [8]. Numerical solutions to the CE consistent with CoE seemed to be necessary. This showed that two non-BFW modes exist but without a more fundamental proof that more modes could not exist. Later analysis showed that there is a maximum of two non-BFW modes since CoE leads to a quadratic equation for wave energy in terms of the energy absorption by scatterers [9]. This paper overcomes the limitations of previous work and derives an analytic solution for the two non-BFW that agrees with previous numerical analysis [8]. Also the symmetry relating the wave properties of a periodic structure with those of its “mirror” structure [9], defined by interchanging the moduli of reflection and transmission coefficients, are obtained analytically in this paper.

Asymmetry of the modulus of reflection and transmission coefficients is shown in this paper to be an even simpler case to solve analytically. The solution is the IEW but with a different wave mode propagating in opposite directions. For elastic scattering, one mode is a passing band, and the other is a stopping band. Hence this asymmetric infinite periodic structure leads to a two-way mirror or wave diode.

2. Conservation of energy in an infinite asymmetric periodic structure

CoE is a universal fundamental principle, but its expression in the case of waves in an infinite asymmetric structure is also a rigid constraint on what wave modes can exist. Because the scatterers define in effect an infinite number of somewhat artificial boundary conditions, CoE is not assured and generally must be separately imposed on solutions to the CE. Wave modes that might seem feasible but do not satisfy the expression for CoE Eqs. (2a,b) below could still exist but only for thin periodic structures or near the boundaries.

The derivation of a formula for CoE applicable to an asymmetric periodic structure is a modification to an existing derivation for a symmetric periodic structure [8]. For the symmetric case, it makes no difference whether an energy source is at $x \to -\infty$ or $x \to \infty$, but the asymmetric case needs to treat separately the two source directions with a different energy flux and periodic structure reflectivity. Also energy absorption by scatterers may depend on SW direction.

Consider an energy source at $x \to -\infty$, and denote the +*x* direction energy flux in the n^{th} cell for the SW as $\left|A_n^{(+)}\right|^2$ and the reverse flux as $\left|B_n^{(-)}\right|^2$. The superscripts $(\pm)$ denote the wavenumbers $\pm k_s$ of the corresponding SW. Time averaging eliminates oscillating flux terms such as those proportional to $A_n^{(+)}B_n^{(-)*}$ etc. Conservation of energy from considering the net fluxes each side of a scatterer at the boundary of the n^{th} and $(n+1)^{\text{th}}$ cells leads to

$$\left|A_{n+1}^{(+)}\right|^2 / \left|A_n^{(+)}\right|^2 = \frac{\sigma^{(+)2} - \left|B_n^{(-)}\right|^2 / \left|A_n^{(+)}\right|^2}{1 - \sigma^{(-)2}\left|B_{n+1}^{(-)}\right|^2 / \left|A_{n+1}^{(+)}\right|^2} \tag{1}$$

where $\sigma^{(+)2}$ is the proportion of energy of a SW with wavenumber $+k_s$ not absorbed by a scatterer, and $\sigma^{(-)2}$ is the energy proportion not absorbed for SW wavenumber $-k_s$. Equation (1) applies everywhere including near boundaries however within a thick periodic structure the cell location is immaterial and Eq. (1) can be rewritten in two forms

$$\bar{\xi}^{(+)} = \frac{\sigma^{(+)2} - \bar{\mu}^{(+)}}{1 - \sigma^{(-)2}\bar{\mu}^{(+)}} \tag{2a}$$

$$\bar{\mu}^{(+)} = \frac{\sigma^{(+)2} - \bar{\xi}^{(+)}}{1 - \sigma^{(-)2}\bar{\xi}^{(+)}} \tag{2b}$$

$\left|A_{n+1}^{(+)}\right|^2 / \left|A_n^{(+)}\right|^2 \to \bar{\xi}^{(+)}$ defines a SW forward energy flux persistence at a scatterer whereas $\left|B_n^{(-)}\right|^2 / \left|A_n^{(+)}\right|^2 \to \left|B_{n+1}^{(-)}\right|^2 / \left|A_{n+1}^{(+)}\right|^2 \to \bar{\mu}^{(+)}$ defines the reflectivity of a periodic structure. Equations (2a,b) exhibit a symmetry between $\bar{\xi}^{(+)}$ and $\bar{\mu}^{(+)}$ which results in related wave properties for two mirror periodic structures, defined by the interchange of their scatterer reflection and transmission coefficients[9]. The changes $\sigma^{(+)} \leftrightarrow \sigma^{(-)}, \bar{\xi}^{(+)} \to \bar{\xi}^{(-)}, \bar{\mu}^{(+)} \to \bar{\mu}^{(-)}$ to Eqs. (2a,b) are CoE for an energy source at $x \to \infty$.

2.1 Incoherent energy wave model for an infinite asymmetric periodic structure

The IEW model, discussed in previous papers [7-9], is easily extended to asymmetric periodic structures and satisfies the CoE Eqs. (2a,b). One equation for $\left|A_n^{(+)}\right|^2$ and $\left|B_n^{(-)}\right|^2$ is derived by relating the flux scattered into the $(n+1)^{\text{th}}$ cell to fluxes incident onto the scatterer from the n^{th} and $(n+1)^{\text{th}}$ cells. A second equation results from the flux scattered into the n^{th} cell. For a source at $x \to -\infty$ and using above definitions of $\bar{\xi}^{(+)}$ and $\bar{\mu}^{(+)}$ these equations are

$$\left|R^{(-)}\right|^2 \bar{\mu}^{(+)}\bar{\xi}^{(+)} - \bar{\xi}^{(+)} + \left|T^{(+)}\right|^2 = 0 \tag{3a}$$

$$\left|T^{(-)}\right|^2 \bar{\mu}^{(+)}\bar{\xi}^{(+)} - \bar{\mu}^{(+)} + \left|R^{(+)}\right|^2 = 0 \tag{3b}$$

where $T^{(\pm)}$ and $R^{(\pm)}$ are transmission and reflection coefficients. Eliminating either $\bar{\xi}^{(+)}$ or $\bar{\mu}^{(+)}$ from Eqs.(3a,b) gives two uncoupled quadratic equations

$$\sigma^{(-)2}\left|T_0^{(-)}\right|^2\bar{\xi}^{(+)2}-\left(1-\sigma^{(+)2}\sigma^{(-)2}\left(1-\left|T_0^{(+)}\right|^2-\left|T_0^{(-)}\right|^2\right)\right)\bar{\xi}^{(+)}+\sigma^{(+)2}\left|T_0^{(+)}\right|^2=0 \tag{4a}$$

$$\sigma^{(-)2}\left|R_0^{(-)}\right|^2\bar{\mu}^{(+)2}-\left(1-\sigma^{(+)2}\sigma^{(-)2}\left(1-\left|R_0^{(+)}\right|^2-\left|R_0^{(-)}\right|^2\right)\right)\bar{\mu}^{(+)}+\sigma^{(+)2}\left|R_0^{(+)}\right|^2=0 \tag{4b}$$

where $\left|T^{(\pm)}\right|=\sigma^{(\pm)}\left|T_0^{(\pm)}\right|$ and $\left|R^{(\pm)}\right|=\sigma^{(\pm)}\left|R_0^{(\pm)}\right|$ such that $\left|T_0^{(\pm)}\right|^2+\left|R_0^{(\pm)}\right|^2=1$. Equations (4a,b) are transformed into each other by the substitutions Eqs. (2a,b), hence satisfy CoE. Equations (4a,b) are symmetrized by defining $\tilde{\xi}=\bar{\xi}^{(+)}\left(\sigma^{(-)}\left|T_0^{(-)}\right|\right)/\left(\sigma^{(+)}\left|T_0^{(+)}\right|\right)$ and $\tilde{\mu}=\bar{\mu}^{(+)}\left(\sigma^{(-)}\left|R_0^{(-)}\right|\right)/\left(\sigma^{(+)}\left|R_0^{(+)}\right|\right)$ leading to

$$\tilde{\xi}^2-2\tilde{\Delta}\tilde{\xi}+1=0 \tag{5a}$$

$$\tilde{\Delta}=1+\frac{\left(1-\tilde{\sigma}^2\right)^2+2\tilde{\sigma}^2\left(1-\tilde{\sigma}^2\right)\left(1-\left|T_0^{(+)}\right|\left|T_0^{(-)}\right|\right)+\tilde{\sigma}^4\left(\left|T_0^{(+)}\right|-\left|T_0^{(-)}\right|\right)^2}{2\tilde{\sigma}^2\left|T_0^{(+)}\right|\left|T_0^{(-)}\right|} \tag{5b}$$

$$\tilde{\mu}^2-2\tilde{\Phi}\tilde{\mu}+1=0 \tag{5c}$$

$$\tilde{\Phi}=1+\frac{\left(1-\tilde{\sigma}^2\right)^2+2\tilde{\sigma}^2\left(1-\tilde{\sigma}^2\right)\left(1-\left|R_0^{(+)}\right|\left|R_0^{(-)}\right|\right)+\tilde{\sigma}^4\left(\left|R_0^{(+)}\right|-\left|R_0^{(-)}\right|\right)^2}{2\tilde{\sigma}^2\left|R_0^{(+)}\right|\left|R_0^{(-)}\right|} \tag{5d}$$

where $\tilde{\sigma}^2=\sigma^{(+)}\sigma^{(-)}$. $\tilde{\xi}$ and $\tilde{\mu}$ are real since $\tilde{\Delta}\geq 1$ and $\tilde{\Phi}\geq 1$. $\bar{\xi}^{(-)}$ and $\bar{\mu}^{(-)}$ for a source at $x\to\infty$ are obtained by swapping superscripts $(+)\leftrightarrow(-)$.

Asymmetric wave propagation and reflection by a periodic structure relate to scatterer properties by $\bar{\xi}^{(-)}/\bar{\xi}^{(+)}=\left(\sigma^{(-)2}\left|T_0^{(-)}\right|^2\right)/\left(\sigma^{(+)2}\left|T_0^{(+)}\right|^2\right)$ and $\bar{\mu}^{(-)}/\bar{\mu}^{(+)}=\left(\sigma^{(-)2}\left|R_0^{(-)}\right|^2\right)/\left(\sigma^{(+)2}\left|R_0^{(+)}\right|^2\right)$. As an example, for elastic scattering $\sigma^{(-)2}=\sigma^{(+)2}=1$ and an extended passing wave in the $+k_s$ direction $\bar{\xi}^{(+)}=1$, $\bar{\mu}^{(+)}=\left|R_0^{(+)}\right|^2/\left|R_0^{(-)}\right|^2<1$ and in the $-k_s$ direction $\bar{\xi}^{(-)}=\left|T_0^{(-)}\right|^2/\left|T_0^{(+)}\right|^2<1$ which by CoE gives $\bar{\mu}^{(-)}=1$ hence a stopped wave. This is a two-way mirror or wave diode although the transmission direction is generally not completely transparent. It is clear however that a periodic structure can achieve a better diode effect than a single asymmetric scatterer.

3. General wave model for an infinite asymmetric periodic structure

The CE for a PSW in an infinite asymmetric periodic structure is derived by the same method as previous papers [7, 8] by applying continuity between two adjacent cells except that the moduli of the complex forward and backward scattering coefficients $T^{(\pm)}$ and $R^{(\pm)}$ respectively, are now different for opposite travelling SW. Assume an energy source at $x\to-\infty$. Denote the amplitude of the $+k_s$ wavenumber SW in any n^{th} cell as $A_n^{(+)}$, then in the $(n+1)^{\text{th}}$ cell it is $A_{n+1}^{(+)}=\gamma^{(+)}A_n^{(+)}$. For the $-k_s$ wavenumber SW the amplitudes are $B_n^{(-)}$ and $B_{n+1}^{(-)}=\gamma^{(-)}B_n^{(-)}$ for the n^{th} and $(n+1)^{\text{th}}$ cells respectively. Continuity considerations at the n^{th} and $(n+1)^{\text{th}}$ cells leads to two homogeneous equations for $A_n^{(+)}$ and $B_n^{(-)}$ that in 2x2 matrix form requires the matrix determinant to be zero. This gives a CE connecting complex PSW amplitudes $\gamma^{(\pm)}$. The CE is soluble by assuming a correlation $\gamma^{(-)}=e^{-i\psi}\gamma^{(+)}$ where phase ψ is later shown to be determined by consistency of the CE with CoE.

Rather than deriving the CE for $\gamma^{(\pm)}$ as a zero determinant it is more general to rewrite the two equations in the form

$$\hat{R}^{(-)}\rho^{(+)}\gamma^{(-)}-\gamma^{(+)}+\hat{T}^{(+)}=0 \tag{6a}$$

$$\hat{T}^{(-)}\rho^{(+)}\gamma^{(-)}-\rho^{(+)}+\hat{R}^{(+)}=0 \tag{6b}$$

where $B_n^{(-)}/A_n^{(+)}\to B_{n+1}^{(-)}/A_{n+1}^{(+)}\to\rho^{(+)}$ and $\hat{T}^{(\pm)}=e^{ik_sd}T^{(\pm)}$, $\hat{R}^{(\pm)}=e^{ik_sd}R^{(\pm)}$. Symmetry considerations lead to defining $\rho^{(-)}=e^{-i\psi}\rho^{(+)}$. Equations (6a,b) are symmetrized by rewriting them in terms of $\tilde{\gamma}=e^{\mp i\psi/2}\sqrt{T^{(-)}/T^{(+)}}\gamma^{(\pm)}$ and $\tilde{\rho}=e^{\mp i\psi/2}\sqrt{R^{(-)}/R^{(+)}}\rho^{(\pm)}$. When decoupled this gives

$$\tilde{\gamma}^2-2\tilde{\Gamma}\tilde{\gamma}+1=0$$
$$\tilde{\Gamma}=\frac{1}{2}\frac{1}{\sqrt{\hat{T}^{(+)}\hat{T}^{(-)}}}\left(e^{i\psi/2}+e^{-i\psi/2}\left(\hat{T}^{(+)}\hat{T}^{(-)}-\hat{R}^{(+)}\hat{R}^{(-)}\right)\right) \tag{7a}$$

$$\tilde{\rho}^2-2\tilde{\Omega}\tilde{\rho}+1=0$$
$$\tilde{\Omega}=\frac{1}{2}\frac{1}{\sqrt{\hat{R}^{(+)}\hat{R}^{(-)}}}\left(e^{i\psi/2}+e^{-i\psi/2}\left(\hat{R}^{(+)}\hat{R}^{(-)}-\hat{T}^{(+)}\hat{T}^{(-)}\right)\right) \tag{7b}$$

Equations (7a,b) under the interchanges $T^{(+)}T^{(-)}\leftrightarrow R^{(+)}R^{(-)}$ exhibit the symmetry $\tilde{\gamma}\leftrightarrow\tilde{\rho}$ which is a generalization of a previous demonstration of the symmetry of mirror structures defined by interchanged transmission and reflection coefficients [9].

The above results arise from considering an energy source at $x\to-\infty$ although the symmetrized Eqs. (7a,b) are independent of the source location and net flux direction. Introducing $\bar{\gamma}^{(+)}=\sqrt{T^{(+)}/T^{(-)}}\tilde{\gamma}$ into Eq. (7a) converts it to an equation for a source at $x\to-\infty$ while $\bar{\gamma}^{(-)}=\sqrt{T^{(-)}/T^{(+)}}(1/\tilde{\gamma})$ gives the equation for a source at $x\to\infty$. Equation (7a) has two solutions $\tilde{\gamma}_{(\pm)}=\tilde{\Gamma}+(\pm)\sqrt{\tilde{\Gamma}^2-1}$ where $\tilde{\gamma}_{(+)}\tilde{\gamma}_{(-)}=1$ so that if one gives $|\tilde{\gamma}_{(+)}|\le 1$ then the other has $1/|\tilde{\gamma}_{(-)}|\le 1$ and applies to a source at $x\to\infty$. The ratio of asymmetric results for the two different sources is $\bar{\gamma}^{(-)}/\bar{\gamma}^{(+)}=T^{(-)}/T^{(+)}$. This analysis when applied to Eq. (7b) gives $\bar{\rho}^{(+)}=\sqrt{R^{(+)}/R^{(-)}}\tilde{\rho}$ and $\bar{\rho}^{(-)}=\sqrt{R^{(-)}/R^{(+)}}(1/\tilde{\rho})$ for the two source directions, and hence the ratio $\bar{\rho}^{(-)}/\bar{\rho}^{(+)}=R^{(-)}/R^{(+)}$.

From Eqs. (7a,b), quadratic equations for the energy flux persistence $\tilde{\xi}=|\tilde{\gamma}|^2$ and reflectivity $\tilde{\mu}=|\tilde{\rho}|^2$ are the same as Eqs. (5a,c) except Eqs. (5b,d) for $\tilde{\Delta}$ and $\tilde{\Phi}$ are replaced by

$$\tilde{\Delta}=1+\left(\sqrt{\left(1-\left(\tilde{\Gamma}'^2+\tilde{\Gamma}''^2\right)\right)^2+4\tilde{\Gamma}''^2}-\left(1-\left(\tilde{\Gamma}'^2+\tilde{\Gamma}''^2\right)\right)\right) \tag{8a}$$

$$\tilde{\Phi}=1+\left(\sqrt{\left(1-\left(\tilde{\Omega}'^2+\tilde{\Omega}''^2\right)\right)^2+4\tilde{\Omega}''^2}-\left(1-\left(\tilde{\Omega}'^2+\tilde{\Omega}''^2\right)\right)\right) \tag{8b}$$

using the notation $q=q'+iq''$ to distinguish real and imaginary parts of any complex q. Extending the above asymmetric wave analysis to asymmetric energy flux analysis leads to $\bar{\xi}^{(+)}=\tilde{\xi}|T^{(+)}|/|T^{(-)}|$, $\bar{\xi}^{(-)}=(1/\tilde{\xi})|T^{(-)}|/|T^{(+)}|$, $\bar{\mu}^{(+)}=\tilde{\mu}|R^{(+)}|/|R^{(-)}|$ and $\bar{\mu}^{(-)}=(1/\tilde{\mu})|R^{(-)}|/|R^{(+)}|$. Then the respective ratios of opposite direction flux persistence and reflectivity are given by $\bar{\xi}^{(-)}/\bar{\xi}^{(+)} = \left|T^{(-)}\right|^2/\left|T^{(+)}\right|^2$ and $\bar{\mu}^{(-)}/\bar{\mu}^{(+)} = \left|R^{(-)}\right|^2/\left|R^{(+)}\right|^2$.

These flux persistence and reflectivity ratios are the same as found for the IEW in Subsect. 2.1. Indeed using CoE Eqs. (2a,b) and their superscript interchanges $(+)\leftrightarrow(-)$ to evaluate ratios $\bar{\xi}^{(-)}/\bar{\xi}^{(+)}$ and $\bar{\mu}^{(-)}/\bar{\mu}^{(+)}$ on the LHS we find linear equations for $\tilde{\Delta}$ and $\tilde{\Phi}$ with solutions identical to IEW Eqs. (5b,d). Provided $\left|T_0^{(+)}\right| \neq \left|T_0^{(-)}\right|$ and hence $\left|R_0^{(+)}\right| \neq \left|R_0^{(-)}\right|$ (asymmetric energy absorption $\sigma^{(+)} \neq \sigma^{(-)}$ makes no difference) then IEW is the unique solution for wave propagation and attenuation in a thick asymmetric periodic structure. IEW must still satisfy the scattering model and this amounts to setting the RHS of Eqs. (8a,b) equal to the RHS of Eqs. (5b,d), and using $\tilde{\Gamma}$ and $\tilde{\Omega}$ from Eqs. (7a,b), gives an equation for ψ. This leads to a linear equation in $\sin(2\zeta)$ and $\cos(2\zeta)$, where $\zeta = k_s d + \tilde{\phi} - \psi/2$, $\tilde{\phi} = \left(\phi^{(+)} + \phi^{(-)}\right)/2$ and $\phi^{(\pm)}$ are SW forward scattering phase shifts for $\pm k_s$. The linear equation transforms to quadratic equations in $\sin(2\zeta)$ and $\cos(2\zeta)$ with two solutions we denote $\zeta_{(\pm)}$. Hence $\psi_{(\pm)} = 2k_s d + 2\left(\tilde{\phi} - \zeta_{(\pm)}\right)$ defines the two correlations of PSW pairs necessary for asymmetric energy propagation from the two possible source directions. For a source at $x \to -\infty$, $\gamma^{(+)} = \left|\gamma^{(+)}\right| e^{i\varphi} e^{ik_s d}$ is the PSW for k_s and $\gamma^{(-)} = e^{-i\psi}\gamma^{(+)} = \left|\gamma^{(+)}\right| e^{-2i\left(\tilde{\phi} - \zeta - \varphi/2\right)} e^{-ik_s d}$ is the PSW for opposite direction $-k_s$. ζ is one of the two solutions $\zeta_{(\pm)}$ that more detailed analysis identifies for the particular source direction.

3.1 Symmetric scattering coefficient magnitudes but asymmetric phase shifts

Whereas asymmetric scattering, where $\left|R_0^{(+)}\right| \neq \left|R_0^{(-)}\right|$ and $\left|T_0^{(+)}\right| \neq \left|T_0^{(-)}\right|$, require exotic designs or metamaterials, symmetric $\left|R_0^{(\pm)}\right| = \left|R_0\right|$ and $\left|T_0^{(\pm)}\right| = \left|T_0\right|$ can coexist with asymmetric phase shifts from geometrically asymmetric scatterers made of conventional materials. For example, complex coefficients $R_0^{(\pm)} = \left|R_0\right| e^{i\chi^{(\pm)}}$ and $T_0^{(\pm)} = \left|T_0\right| e^{i\phi^{(\pm)}}$ can be derived for an asymmetric refractive scatterer using well-known theory [10]. This asymmetry may be two different internal materials with different thicknesses and impedances, and noncentral geometry such as a sandwich of two or more layers. The simplest case is two layers and no energy absorption. The forward scattering phase shifts have opposite signs for opposite SW wavenumbers i.e. $\phi^{(\pm)} = \pm\phi$. The reflection phase shifts have the form $\chi^{(+)} = \phi + \varepsilon, \chi^{(-)} = -\phi + \varepsilon + (\pm)\pi$ with part of these phase shifts ε that does not change sign for opposite SW directions. Then the phase shift difference arising in PSW theory is $\delta = \left(\left(\chi^{(-)} + \chi^{(+)}\right) - \left(\phi^{(-)} + \phi^{(+)}\right)\right)/2 = \varepsilon + (\pm)\pi/2$ where ε=0 holds for a geometrically symmetric scatterer.

Unlike the asymmetric case $\left|R_0^{(+)}\right| \neq \left|R_0^{(-)}\right|$ and $\left|T_0^{(+)}\right| \neq \left|T_0^{(-)}\right|$, symmetric $\left|R_0^{(\pm)}\right| = \left|R_0\right|$ and $\left|T_0^{(\pm)}\right| = \left|T_0\right|$ do not have just one PSW solution with a unique $\tilde{\Delta}$ and $\tilde{\Phi}$. IEW is a solution but so is the BFW where ε=0. Two other PSW solutions, dubbed non-BFW here, also exist and are discussed to some extent in previous papers [8, 9]. Unlike IEW and BFW, the CE for the non-BFW is not redundant with CoE which must be introduced explicitly as a constraint on the CE solutions. In reference [8] the non-BFW case is solved numerically however $\tilde{\Delta}$ and $\tilde{\Phi}$ can be derived analytically as shown below.

To facilitate combining CoE with the CE, after using some above relationships for $\tilde{\rho}$, $\tilde{\mu}$, $\tilde{\gamma}$ and $\tilde{\xi}$, Eqs. (2b) and (6a) lead to two equations for $\bar{\mu}^{(+)}$

$$\overline{\mu}^{(+)} = \sigma^{(+)2} \frac{1 - \frac{1}{\tilde{\sigma}^2}\tilde{\xi}}{1 - \tilde{\sigma}^2\tilde{\xi}} \tag{9a}$$

$$\overline{\mu}^{(+)} = \frac{1}{\sigma^{(-)2}|R_0|^2}\left(1 + \frac{|T_0|^2}{\tilde{\xi}} - \frac{2|T_0|}{1-\tilde{\xi}^2}\left(\left(e^{i\zeta}\tilde{\Gamma} + e^{-i\zeta}\tilde{\Gamma}^*\right) - \tilde{\xi}\left(e^{-i\zeta}\tilde{\Gamma} + e^{i\zeta}\tilde{\Gamma}^*\right)\right)\right) \tag{9b}$$

Then substituting the RHS of Eq. (9a) into the LHS of Eq. (9b) gives an equation for unknowns $\tilde{\xi}$ and ζ . This equation is fourth order in $\tilde{\xi}$ but it can be reduced to second order using power reduction by $\tilde{\xi}^2 = 2\tilde{\Delta}\tilde{\xi} - 1$ from Eq. (5a). This results in a quadratic equation

$$F\tilde{\xi}^2 - 2G\tilde{\xi} + H = 0 \tag{10}$$

where *F*, *G* and *H* are functions of the scattering parameters and unknown ζ . The two solutions of Eq. (10) for $\tilde{\xi}$, we denote $\tilde{\xi}_{(\pm)}$, must have the same property $\tilde{\xi}_{(+)}\tilde{\xi}_{(-)} = 1$ as Eq. (5a) and so requires $F = H$. Then consistency of Eqs. (5a) and (10) unambiguously determines that $\tilde{\Delta} = G/H$. Inserting the functions *G* and *H* we find that $\tilde{\Delta}$ cancels out leaving the equation for ζ

$$\left(1 - \tilde{\sigma}^2\right)\cos(\zeta)\tilde{\Gamma}' - \left(1 + \tilde{\sigma}^2\right)\sin(\zeta)\tilde{\Gamma}'' = \frac{1}{2\tilde{\sigma}|T_0|}\left(1 - \tilde{\sigma}^4\right) \tag{11}$$

Equation (11) is a generalization of a previously derived requirement $\tilde{\Gamma}'' = 0$ of elastic scattering $\tilde{\sigma} = 1$ [9]. The two phases $\zeta_{(\pm)}$, corresponding to the two non-BFW modes, are found from

$$\begin{aligned} \sin\left(2\zeta_{(\pm)}\right) &= -\tilde{\sigma}^2\cos(\delta)\sin(\delta) - (\pm)\cos(\delta)\sqrt{1 - \tilde{\sigma}^4\cos^2(\delta)} \\ \cos\left(2\zeta_{(\pm)}\right) &= \tilde{\sigma}^2\cos^2(\delta) - (\pm)\sin(\delta)\sqrt{1 - \tilde{\sigma}^4\cos^2(\delta)} \end{aligned} \tag{12}$$

Equations (11) and (12) can be used to derive $\tilde{\Delta}$ and $\tilde{\Phi}$ from Eqs. (8a,b). An alternative is to use $F = H$ that leads to a quadratic equation for $\tilde{\Delta}$

$$\tilde{\Delta}^2 - 2U\tilde{\Delta} + V = 0 \tag{13a}$$

to be compared with Eq. (8a) where $\tilde{\Delta}$ satisfies the quadratic equation

$$\tilde{\Delta}^2 - 2\left(\tilde{\Gamma}'^2 + \tilde{\Gamma}''^2\right)\tilde{\Delta} + 2\left(\tilde{\Gamma}'^2 - \tilde{\Gamma}''^2\right) - 1 = 0 \tag{13b}$$

Then eliminating $\tilde{\Delta}^2$ from Eqs. (13a,b) gives

$$\tilde{\Delta} = 1 + \frac{1}{2}\frac{1 + V - 2U + 4\tilde{\Gamma}''^2}{U - \left(\tilde{\Gamma}'^2 + \tilde{\Gamma}''^2\right)} \tag{14a}$$

$\tilde{\Phi}$ can be derived from $\tilde{\Delta}$ using the relation

$$\tilde{\Phi} = 1 + \frac{\left(1 - \tilde{\sigma}^2\right)^2\left(\tilde{\Delta} + 1\right)}{2\tilde{\sigma}^2\left(\tilde{\Delta} - 1\right) - \left(1 - \tilde{\sigma}^2\right)^2} \tag{14b}$$

that comes from the symmetry $\tilde{\xi} \leftrightarrow \tilde{\mu}$ of CoE. Then the two non-BFW modes correspond to

$$\tilde{\Delta}_{(\pm)} = 1 + \frac{\left(1 - \tilde{\sigma}^2\right)^2 + 2\tilde{\sigma}^2\left(1 - \tilde{\sigma}^2\cos^2(\delta) - (\pm)\sin(\delta)\sqrt{1 - \tilde{\sigma}^4\cos^2(\delta)}\right)|R_0|^2}{2\tilde{\sigma}^2|T_0|^2} \tag{15a}$$

$$\tilde{\Phi}_{(\pm)} = 1 + \frac{\left(1-\tilde{\sigma}^2\right)^2 + 2\tilde{\sigma}^2\left(1-\tilde{\sigma}^2\cos^2(\delta) + (\pm)\sin(\delta)\sqrt{1-\tilde{\sigma}^4\cos^2(\delta)}\right)\left|T_0\right|^2}{2\tilde{\sigma}^2\left|R_0\right|^2} \quad (15b)$$

Thus there are two non-BFW solutions $\tilde{\xi}_{(\pm)}$ and $\tilde{\mu}_{(\pm)}$ for both possible source directions. For elastic scattering $\tilde{\sigma}=1$ and $\sin(\delta)\neq 0$ Eqs. (15a,b) show that one mode is a passing band and the other a stopping band. The two non-BFW modes coincide with each other and IEW for $\sin(\delta)=0$, and they are all stopping bands in the case of elastic scattering. Equations (15a,b) also show that the interchange $\left|T_0\right|^2 \leftrightarrow \left|R_0\right|^2$ corresponds to $\tilde{\Delta}_{(\pm)} \leftrightarrow \tilde{\Phi}_{(\mp)}$, which explains why $\tilde{\mu}$ in one structure equates to $\tilde{\xi}$ in its mirror structure [9].

4. Summary

This paper shows that the wave properties of an infinite 1D periodic structure depend on the symmetry of the identical, equally spaced scatterers. Fully symmetric SW scatterers give rise to $\varepsilon=0, \psi=0$ and hence BFW which oscillate with SW wavenumber between passing and stopping bands for elastic scattering. For the same scattering parameters that allow BFW, the IEW and two non-BFW are also possible through correlated pairs of SW coupled by scattering such that $\psi \neq 0$.

When the internal structure of a scatterer is asymmetric such that the difference of backward and forward phase shifts δ deviate from $\pm\pi/2$, but the modulus of the reflection and transmission coefficients are the same in both SW directions, inconsistency with CoE eliminates BFW but the IEW and two non-BFW are still possible and propagate equally in both directions. For elastic scattering, one non-BFW is a passing band and the other is a stopping band. For the special case of a scattering phase shift difference $\delta = 0, \pm\pi$, the IEW and two non-BFW coincide.

The theory for non-BFW requires simultaneous solutions for the CE and CoE that must be explicitly coupled. This contrasts with the BFW and IEW where the CE is redundant with the CoE. For asymmetric moduli of scatterer reflection and transmission coefficients, the IEW is the only solution for a thick periodic structure. For asymmetric elastic scattering, one IEW mode is a passing band while the other IEW mode for the opposite direction is a stopping band, essentially a wave diode or two-way mirror. Such an asymmetric periodic structure might be a practical way to strongly attenuate wave propagation in a particular direction using many scatterers even when each one has a small reflection coefficient and small degree of individual nonreciprocity.

REFERENCES

1 Scalora, M., Dowling, J. P., Bowden, C.M., Bloemer, M. J. The photonic band edge optical diode, *Journal of Applied Physics*, **76** (4), 2023-2026, (1994)

2 Miroshnichenko, A.E., Brasselet, E. and Kivshar, Y.S. Reversible optical nonreciprocity in periodic structures with liquid crystals, *Applied Physics Letters*, **96**, 063302, (2010)

3 Fleury, R., Sounas, D. L., Haberman, M. R., Alù, A. Nonreciprocal Acoustics, *Acoustics Today*, **11** (3), 14 – 21, (2015)

4 Haberman, M. R., and Norris, A. **N.** Acoustic Metamaterials, *Acoustics Today*, **12** (3), 31 – 39, (2016)

5 Brillouin, L. *Wave Propagation in Periodic Structures,* Second Edition, Dover Publications, Inc., (1953).

6 Hwang, R. *Periodic Structures: Mode-Matching Approach and Applications in Electromagnetic Engineering*, John Wiley and Sons, Singapore, Chap. 3, (2013).

7 McMahon, D. Wave propagation in infinite periodic structures taking into account energy absorption, *Proceedings of Acoustics 2015*, Hunter Valley, Australia, 15-18 November, (2015).

8 McMahon, D. Wave propagation in infinite periodic structures, *Proceedings of the 23rd International Congress on Sound and Vibration* (ICSV23), Athens, Greece, (2016).

9 McMahon, D. Symmetry of waves in an infinite 1D periodic structure under the interchange of forward and backward scattering coefficients, *Proceedings of Acoustics 2016*, Brisbane, Australia, 9-11 November, (2016).

10 Kinsler, L. E., Frey, A. R., Coppens, A. B. and Sanders, J. V., *Fundamentals of Acoustics,* Fourth Edition, John Wiley and Sons, Chap. 6, (1999).