25th International Congress on Sound and Vibration
8-12 July 2018 HIROSHIMA CALLING
ICSV25

COHERENT AND INCOHERENT WAVES WITHIN A ONE DIMENSIONAL PERIODIC STUCTURE OF POINT SCATTERERS INCLUDING BOUNDARIES

Darryl McMahon

Curtin University, Centre for Marine Science and Technology, Bentley, WA 6102, Australia
email: D.Mcmahon@curtin.edu.au

Periodic structure waves (PSW) for an infinite 1D periodic structure of equally spaced point structure wave (SW) scatterers were previously derived from considering continuity and energy flux equations at any two adjacent cells where PSW were defined to be identical in all cells. This paper uses a 2x2-scattering matrix method where previous results for periodic structures are reproduced but also allows boundaries for finite structures to be included. The matrix method shows there are more possible PSW than the cell independent PSW assumed for infinite periodic structures. Indeed cell independent PSW are just the eigenvectors of the scattering matrix. The smallest possible finite periodic structure with just two scatterers is analysed in some detail and shows more simply than for an infinite periodic structure the physical constraints on scattering models imposed by conservation of energy (CoE). This confirms the previous conclusion from infinite 1D periodic structures that the only continuous wideband coherent PSW is the Bloch-Floquet wave (BFW) for symmetric scatterers. Asymmetric scatterers, including those that exhibit nonreciprocal wave propagation, allow coherent PSW at certain discrete wavenumbers but more generally, consistency with CoE requires PSW to be incoherent over continuous bandwidths.

Keywords: wave, periodic, scattering, coherent, boundary

1. Introduction

Wave propagation in periodic structures has a long history starting the nineteenth century with the development of simple models and wave equations for media with spatially sinusoidal phase speeds [1]. Later developments of x-ray scattering and solid-state electronics exposed the practical need to understand wave phenomena for periodic structures. A key concept for electron waves in solids is the Bloch theorem for the wavefunction and shows that electrons can exist at certain energy bands but not other “forbidden” bands (also called band-gaps) [2,3]. Similar concepts apply to the macroscopic periodic systems of structural engineering but with freedom to design structures with desirable properties such as to strongly attenuate wave propagation over certain frequency bands [4].

For macroscopic systems, it is feasible to design components of a periodic structure that may not satisfy the assumptions for linear wave theory, particularly nonreciprocal wave propagation [5]. Nonreciprocal propagation by a periodic structure of asymmetric scatterers was found to enhance the wave diode effect and achieves greater attenuation than symmetric scatterers [6]. The use of scattering theory has had the advantage of analytic results for 1D periodic structures [7-10]. To derive wave phenomena for asymmetric periodic structures, the Bloch theorem often used as a starting point was not assumed. Scattering theory applied to periodic structures, and crucially coupled with the explicit constraint of CoE, recovers known coherent BFW for symmetric scatterers but also showed that coherent PSW over continuous bandwidths are inconsistent with asymmetric scatterers [7, 8, 10].

For clarity, this paper defines this distinction between coherent and incoherent wave effects. The coherence that BFW exhibit is the well-known wavenumber dependent passing bands and stopping bands of wavenumber widths proportional to $1/d$ where d is the spacing between scatterers. The absence of band structures proportional to $1/d$ is considered to be incoherence which in this paper are an infinite bandwidth (i.e. inversely proportional to the width of the point scatterers).

This paper extends previous work to better demonstrate the division between coherent and incoherent wave phenomena, and include finite periodic structures where boundaries are considered. Section 2 summarises the 2x2-matrix method for waves in a finite periodic structure. Section 3 considers reflection and transmission by a finite periodic structure, including an analysis in some detail of the smallest possible periodic structure of just two scatterers. This shows that whereas a single scatterer conforms to CoE, two such scatterers any distance d apart only conform to CoE for a coherent wave over continuous wave bands when scatterers are symmetric (i.e. same conditions as for BFW). Two scatterers reaffirm that energy propagation by asymmetric periodic structures over continuous bandwidths does not depend on the spacing d, effectively requiring incoherent wave propagation. Possible physical concepts for this incoherence are discussed. Section 4 summarises the 2x2-matrix method for energy propagation in finite 1D periodic structures, which applies equally to coherent and incoherent waves.

2. PSW in a finite asymmetric periodic structure

An increasing cell index n is used to indicate cells at increasing position x. Superscripts $(\pm)$ are used to distinguish the two possible directions of SW propagation incident onto and within a generally asymmetric finite periodic structure. Denote the amplitude of the $+k_s$ wavenumber SW in any n^{th} cell as $A_n^{(+)}$ at the position x_{n-1} of the $(n\text{-}1)^{\text{th}}$ scatterer. Then in the $(n+1)^{\text{th}}$ cell the amplitude is $A_{n+1}^{(+)}$ at the position x_n of the n^{th} scatterer. For the $-k_s$ wavenumber SW the amplitudes are $B_n^{(-)}$ and $B_{n+1}^{(-)}$ in the n^{th} and $(n+1)^{\text{th}}$ cells at scatterer positions position x_n and x_{n+1} respectively. An advantage of defining $A_n^{(+)}$ and $B_n^{(-)}$ at scatterers separated a spacing d apart is that phase factors e^{ik_sd} can be grouped with the forward transmission coefficients $T^{(\pm)}$ and backward reflection coefficients $R^{(\pm)}$ as in $\hat{T}^{(\pm)} = e^{ik_sd}T^{(\pm)}$, $\hat{R}^{(\pm)} = e^{ik_sd}R^{(\pm)}$. This is valid even if there is only one scatterer where d is arbitrary and the phase factor cancels out of scattering results.

A single scatterer is defined by a 2x2 scattering matrix $\mathbf{M}$ that transitions a vector $\begin{pmatrix} A_n^{(+)} & B_n^{(-)} \end{pmatrix}^T$ by

$$\begin{pmatrix} A_{n+1}^{(+)} \\ B_{n+1}^{(-)} \end{pmatrix} = \mathbf{M} \begin{pmatrix} A_n^{(+)} \\ B_n^{(-)} \end{pmatrix} \tag{1}$$

The elements of $\mathbf{M}$ are denoted $M_{AA}, M_{AB}, M_{BA}, M_{BB}$. Previous papers for infinite periodic structures make the simplifying assumption that these vectors are cell independent as defined by

$$\begin{pmatrix} A_{n+1}^{(+)} \\ B_{n+1}^{(-)} \end{pmatrix} = \begin{pmatrix} \gamma A_n^{(+)} \\ \gamma B_n^{(-)} \end{pmatrix} \tag{2}$$

Combining Eqs.(1) and (2) gives

$$\begin{pmatrix} \gamma - M_{AA} & -M_{AB} \\ -M_{BA} & \gamma - M_{BB} \end{pmatrix} \begin{pmatrix} A_n^{(+)} \\ B_n^{(-)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3a}$$

$$\det \begin{pmatrix} \gamma - M_{AA} & -M_{AB} \\ -M_{BA} & \gamma - M_{BB} \end{pmatrix} = 0 \tag{3b}$$

Equation (3b) is the characteristic equation (CE) for $\mathbf{M}$

$$\begin{gathered} \gamma^2 - 2\Gamma\gamma + \det(\mathbf{M}) = 0 \\ \Gamma = \frac{1}{2}\left(M_{AA} + M_{BB}\right) \end{gathered} \tag{4a}$$

$$\gamma^{(\pm)} = \frac{1}{2}\left(M_{AA} + M_{BB}\right) - (\pm)\sqrt{\frac{1}{4}\left(M_{AA} + M_{BB}\right)^2 - \det(\mathbf{M})}, \quad \det(\mathbf{M}) = \gamma^{(+)}\gamma^{(-)} \tag{4b}$$

$$\mathbf{U}^{(\pm)} = \begin{pmatrix} U_A^{(\pm)} \\ U_B^{(\pm)} \end{pmatrix} = U_A^{(\pm)} \begin{pmatrix} 1 \\ \dfrac{\gamma^{(\pm)} - M_{AA}}{M_{AB}} \end{pmatrix} \tag{4c}$$

where the two solutions to Eq. (4a) for γ are the eigenvalues $\gamma^{(\pm)}$ of $\mathbf{M}$, corresponding to eigenvectors $\mathbf{U}^{(\pm)}$ where $U_A^{(\pm)}$ are determined by the sources but otherwise are arbitrary. The signs in Eq. (4b) give $\gamma^{(+)} < \gamma^{(-)}$ so that for a single source at $x \to -\infty$ the backward SW components $B_n^{(-)}$ decrease with increasing *n* whereas for a single source at $x \to \infty$ the forward SW components $A_n^{(+)}$ increase with increasing *n*.

Scatterer asymmetry is embodied by $\det(\mathbf{M}) \neq 1$ for which the two directions are not equivalent. Equations (4a, b, c) are symmetrized by defining

$$\tilde{\gamma} = \frac{1}{\sqrt{\det(\mathbf{M})}}\gamma, \quad \tilde{\gamma}^{(\pm)} = \frac{1}{\sqrt{\det(\mathbf{M})}}\gamma^{(\pm)}, \quad \tilde{\Gamma} = \frac{1}{\sqrt{\det(\mathbf{M})}}\Gamma \tag{5}$$

Since the output vector $\begin{pmatrix} A_{n+1}^{(+)} & B_{n+1}^{(-)} \end{pmatrix}^T$ of the n^{th} scatterer is the input vector for the $n+1^{\text{th}}$ scatterer, Eq. (1) is generalised to *m* equally spaced and identical scatterers by

$$\begin{pmatrix} A_{n+m}^{(+)} \\ B_{n+m}^{(-)} \end{pmatrix} = \mathbf{M}^m \begin{pmatrix} A_n^{(+)} \\ B_n^{(-)} \end{pmatrix} \tag{6}$$

When *m* is large, rather than multiply *m* times the elements of $\mathbf{M}$, it is more efficient to derive $\mathbf{M}^m$ from the two eigenvalues $\gamma^{(\pm)}$ noting the eigenvectors $\mathbf{U}^{(\pm)}$ of $\mathbf{M}$ are also eigenvectors of $\mathbf{M}^m$. This leads to

$$\mathbf{M}^m = \frac{\left(\gamma^{(-)m} - \gamma^{(+)m}\right)}{\left(\gamma^{(-)} - \gamma^{(+)}\right)}\mathbf{M} - \gamma^{(-)}\gamma^{(+)}\frac{\left(\gamma^{(-)m-1} - \gamma^{(+)m-1}\right)}{\left(\gamma^{(-)} - \gamma^{(+)}\right)}\mathbf{I} \tag{7}$$

where $\mathbf{I}$ is the unit matrix. The determinant of the product of two matrices is the product of their determinants so $\det(\mathbf{M}^m) = \det(\mathbf{M})^m$ which is satisfied by Eq. (7).

It is possible for the vectors $\begin{pmatrix} A_n^{(+)} & B_n^{(-)} \end{pmatrix}^T, n=1,..m$ are eigenvectors for a finite periodic structure of m scatterers but because Eq. (4c) shows both $A_n^{(+)}$ and $B_n^{(-)}$, n=1,.m are nonzero either an energy source is needed at both ends or a reflector at one end in the case of one source. A single source can produce cell independent eigenvector solutions for an infinite or semi-infinite periodic structure. More generally a single SW source creates the initial vector $\begin{pmatrix} A_1^{(+)} & B_1^{(-)} \end{pmatrix}^T$ that is a superposition of the eigenvectors but the relative coefficients of the superposition varies with n i.e. such vectors are not cell independent.

3. CoE constraints on multiple asymmetric scatterers

Suppose n=1 is the first scatterer of a finite periodic structure of m scatterers, and $A_1^{(+)}$ is the SW amplitude from a source at $x < x_1$ but there is no source at $x > x_m$ (i.e. $B_{1+m}^{(-)} = 0$). Then the forward transmitted $A_{1+m}^{(+)}$ and backward reflected $B_1^{(-)}$ are uniquely determined by $A_1^{(+)}$. From Eq. (6)

$$\begin{aligned} A_{m+1}^{(+)} &= M_{AA}^{(m)} A_1^{(+)} + M_{AB}^{(m)} B_1^{(-)} \\ B_{m+1}^{(-)} &= M_{BA}^{(m)} A_1^{(+)} + M_{BB}^{(m)} B_1^{(-)} = 0 \end{aligned} \tag{8a}$$

$$\begin{aligned} A_{m+1}^{(+)} &= \frac{\det\left(\mathbf{M}^{(m)}\right)}{M_{BB}^{(m)}} A_1^{(+)} \\ B_1^{(-)} &= -\frac{M_{BA}^{(m)}}{M_{BB}^{(m)}} A_1^{(+)} \end{aligned} \tag{8b}$$

The superscript for $\mathbf{M}^{(m)} \equiv \mathbf{M}^m$ is used to indicate that its elements are not the m^{th} powers of M_{AA}, M_{AB}, M_{BA} & M_{BB}.

Consider elastic scattering for which CoE is easiest tested. CoE is then $\left|A_{m+1}^{(+)}\right|^2 + \left|B_1^{(-)}\right|^2 = \left|A_1^{(+)}\right|^2$ which is satisfied if the elements of $\mathbf{M}^{(m)}$ have the property

$$\left|M_{BB}^{(m)}\right|^2 = \left|\det\left(M^{(m)}\right)\right|^2 + \left|M_{BA}^{(m)}\right|^2 \tag{9}$$

3.1 Elastic scattering by two identical asymmetric scatterers

For two identical scatterers

$$\mathbf{M}^{(2)} = \mathbf{M}^2 = \begin{pmatrix} M_{AA}^2 + M_{AB} M_{BA} & M_{AB}\left(M_{AA} + M_{BB}\right) \\ M_{BA}\left(M_{AA} + M_{BB}\right) & M_{BB}^2 + M_{AB} M_{BA} \end{pmatrix} \tag{10}$$

CoE for elastic scattering from Eq. (9) for m=2 is

$$\left|M_{BB}^2 + M_{AB} M_{BA}\right|^2 = \left|\det\left(\mathbf{M}\right)\right|^4 + \left|M_{BA}\left(M_{AA} + M_{BB}\right)\right|^2 \tag{11}$$

The elements of $\mathbf{M}$ for a single scatterer have the general asymmetric form [10]

$$M_{AA} = \frac{1}{\hat{T}^{(-)}}\left(\hat{T}^{(+)}\hat{T}^{(-)} - \hat{R}^{(+)}\hat{R}^{(-)}\right), \quad M_{AB} = \frac{\hat{R}^{(-)}}{\hat{T}^{(-)}}, \quad M_{BA} = -\frac{\hat{R}^{(+)}}{\hat{T}^{(-)}}, \quad M_{BB} = \frac{1}{\hat{T}^{(-)}} \tag{12}$$

Consider elastic scattering where the transmission and reflection coefficient magnitudes are symmetric i.e. $\left|T^{(+)}\right|=\left|T^{(-)}\right|=\left|T_0\right|$, $\left|R^{(+)}\right|=\left|R^{(-)}\right|=\left|R_0\right|$, $\left|T_0\right|^2+\left|R_0\right|^2=1$ but the forward and backward scattering phase shifts $\phi^{(\pm)}$ and $\chi^{(\pm)}$ respectively are asymmetric. $\mathbf{M}$ is then given by

$$\mathbf{M}=e^{i\left(\phi^{(+)}-\phi^{(-)}\right)/2}\begin{pmatrix} e^{i\zeta}\dfrac{1}{\left|T_0\right|}\left(\left|T_0\right|^2-e^{2i\delta}\left|R_0\right|^2\right) & e^{-i\left(\chi^{(+)}-\chi^{(-)}\right)/2}e^{i\delta}\dfrac{\left|R_0\right|}{\left|T_0\right|} \\ -e^{i\left(\chi^{(+)}-\chi^{(-)}\right)/2}e^{i\delta}\dfrac{\left|R_0\right|}{\left|T_0\right|} & e^{-i\zeta}\dfrac{1}{\left|T_0\right|} \end{pmatrix},\quad \det\left(\mathbf{M}\right)=e^{i\left(\phi^{(+)}-\phi^{(-)}\right)} \tag{13}$$

$$\zeta=k_s d+\left(\phi^{(+)}+\phi^{(-)}\right)/2,\quad \delta=\left(\chi^{(+)}+\chi^{(-)}\right)/2-\left(\varphi^{(+)}+\varphi^{(-)}\right)/2$$

Equation (13) satisfies Eq. (9) for m=1 so a single isolated asymmetric scatterer is consistent with CoE. Evaluating the LHS and RHS of Eq. (11) using Eq. (13) gives

$$\left|M_{BB}^2+M_{AB}M_{BA}\right|^2=1+4\frac{\left|R_0\right|^2}{\left|T_0\right|^4}\sin^2\left(\zeta+\delta\right) \tag{14a}$$

$$\left|\det\left(\mathbf{M}\right)\right|^4+\left|M_{BA}\left(M_{AA}+M_{BB}\right)\right|^2=1+4\frac{\left|R_0\right|^2}{\left|T_0\right|^4}\sin^2\left(\zeta+\delta\right)+4\frac{\left|R_0\right|^2}{\left|T_0\right|^2}D \tag{14b}$$

$$D=\cos\left(\delta\right)\left(\cos\left(2\zeta+\delta\right)-\left|R_0\right|^2\cos\left(\delta\right)\right) \tag{14c}$$

CoE requires $D=0$. The solution $\cos\left(\delta\right)=0,\ \delta=\pm\pi/2$ independent of wavenumber corresponds to symmetric scattering and coincides with the condition for BFW in an infinite periodic structure. The other solution $D=0,\ \cos\left(\delta\right)\neq 0$ can hold for certain discrete wavenumbers since ζ depends on spacing d independent of asymmetric phase shift parameter δ. More generally, at any other wavenumber, two asymmetric scatterers, even if they satisfy CoE when isolated, do not satisfy CoE as a pair of interacting scatterers. A potential solution to this problem introduced in previous papers for infinite asymmetric periodic structures [8, 10] is applied and discussed further in Section 3.2.

3.2 Incoherent waves between two asymmetric scatterers

Previous work showed that asymmetric scatterers in an infinite periodic structure satisfy CoE if the backward wave is not only a result of scattering but introduces and additional phase shift ψ where Eq. (2) is modified to $B_{n+1}^{(-)}=e^{-i\psi}\gamma B_n^{(-)}$. This is equivalent to transforming Eqs. (13) and (14a, b, c) by $\zeta\rightarrow\zeta-\psi/2=\theta/2$ where $\psi/2$ cancels the $k_s d$ term in phase ζ and makes the phase difference θ independent of the spacing d between scatterers. One possibility is the offset of δ from $\pm\pi/2$ causes phase randomisation from multiple reflections between scatterers. In any case ψ dependence on d cannot be a property of individual scatterers but must be a wave decorrelation phenomenon for pairs of scatterers. Then energy transfer between a pair of asymmetric scatterers does not exhibit the wavenumber dependent maxima and minima of symmetric scatterers ($\delta=\pm\pi/2$), and by the definition in Section 1 is incoherent.

The additional phase shift ψ is equivalent to modifying $\mathbf{M}$ to $\bar{\mathbf{M}}$ where

$$\bar{\mathbf{M}} = \begin{pmatrix} e^{-i\psi/2} M_{AA} & M_{AB} \\ M_{BA} & e^{i\psi/2} M_{BB} \end{pmatrix}$$
$$\det\left(\bar{\mathbf{M}}\right) = \det\left(\mathbf{M}\right) = e^{i\left(\phi^{(+)} - \phi^{(-)}\right)} \tag{15a}$$

leading to the CE for $\bar{\mathbf{M}}$

$$\bar{\gamma}^2 - 2\bar{\Gamma}\bar{\gamma} + \det\left(\bar{\mathbf{M}}\right) = 0$$
$$\bar{\gamma} = e^{-i\psi/2}\gamma \tag{15b}$$
$$\bar{\Gamma} = \frac{1}{2}\left(e^{-i\psi/2} M_{AA} + e^{i\psi/2} M_{BB}\right)$$

Essentially the same transformation has been applied to infinite asymmetric periodic structures [10].

The CoE requirement $D = 0,\ \cos\left(\delta\right) \neq 0$ of Eq. (14c) becomes

$$\cos\left(\theta + \delta\right) = \left|R_0\right|^2 \cos\left(\delta\right) \tag{16a}$$

with two solutions

$$\cos\left(\theta_\pm\right) = \left|R_0\right|^2 \cos^2\left(\delta\right) + \left(\pm\right)\sin\left(\delta\right)\sqrt{1 - \left|R_0\right|^4 \cos^2\left(\delta\right)}$$
$$\sin\left(\theta_\pm\right) = -\left|R_0\right|^2 \sin\left(\delta\right)\cos\left(\delta\right) + \left(\pm\right)\cos\left(\delta\right)\sqrt{1 - \left|R_0\right|^4 \cos^2\left(\delta\right)} \tag{16b}$$

The ratio of forward transmitted and incident fluxes has two possible values

$$\frac{\left|A_3^{(+)}\right|^2}{\left|A_1^{(+)}\right|^2} = \frac{1}{1 + 4\dfrac{\left|R_0\right|^2}{\left|T_0\right|^4}\sin^2\left(\dfrac{\theta_\pm}{2} + \delta\right)} \tag{17a}$$

$$\frac{\left|A_3^{(+)}\right|^2}{\left|A_1^{(+)}\right|^2} = \frac{1}{1 + 2\dfrac{\left|R_0\right|^2}{\left|T_0\right|^4}\left(1 - \left|R_0\right|^2 \cos^2\left(\delta\right) + \left(\pm\right)\sin\left(\delta\right)\sqrt{1 - \left|R_0\right|^4 \cos^2\left(\delta\right)}\right)} \tag{17b}$$

For $\lim \delta \to \pm\pi/2$ these two solutions for asymmetric incoherent scattering match the maxima and minima for coherent symmetric scattering with $\delta = \pm\pi/2$ but have infinite bandwidths and do not oscillate with wavenumber. The same effect is found for the two non-BFW solutions of an asymmetric infinite periodic structure and BFW for a symmetric structure [6]. Unsolved is a more detailed theory for asymmetric scattering that can give a physical origin for cancelling phase ψ and answer what asymmetric scatterer properties determine which of the two solutions apply. Interestingly for $d \to 0$ coherent scattering itself becomes incoherent and using the property $\phi^{(+)} + \phi^{(-)} = 0$ of refractive media [6] coincides with the incoherent solution with the smallest forward scattering. Note that for asymmetric magnitudes of the scattering coefficients $\left|T^{(+)}\right| \neq \left|T^{(-)}\right|$ and $\left|R^{(+)}\right| \neq \left|R^{(-)}\right|$ the PSW is found to be uniquely the incoherent energy wave (IEW) model [10].

4. Energy propagation in a finite 1D periodic structure

Denote the $+x$ direction energy flux[1] in the n^{th} cell for the SW as $\left|A_n^{(+)}\right|^2$ and the reverse flux as $\left|B_n^{(-)}\right|^2$. These fluxes transition at scatterers according to a 2x2 asymmetric scattering matrix $\mathbf{E}$ and hence can be applied to a finite periodic structure. Since energy fluxes are incoherent, CoE satisfied for a single scatterer (symmetric or asymmetric) guarantees that CoE is also satisfied for any *m* scatterers. The elements of $\mathbf{E}$ are interrelated by CoE similar to Eq. (9).

Two functions Δ and Ω arise in the CE for the eigenvalues ξ (energy flux persistence) for $\mathbf{E}$ and μ (energy flux reflectivity) for a matrix $\mathbf{Z}$ that is derived from $\mathbf{E}$ using a relationship between ξ and μ from CoE [6, 10]. From the matrix $\mathbf{E}$ the CE for ξ is

$$\begin{aligned}
&\det\begin{pmatrix} \xi - E_{AA} & -E_{AB} \\ -E_{BA} & \xi - E_{BB} \end{pmatrix} = 0 \\
&\xi^2 - 2\Delta\xi + \det(\mathbf{E}) = 0 \\
&\Delta = \frac{1}{2}\left(E_{AA} + E_{BB}\right)
\end{aligned} \tag{18}$$

The CE for ξ can be transformed to this CE for μ

$$\begin{aligned}
&\det\begin{pmatrix} \mu + \dfrac{E_{AA}}{E_{AB}} & -\dfrac{1}{E_{AB}} \\ \dfrac{1}{E_{AB}}\det(\mathbf{E}) & \mu - \dfrac{E_{BB}}{E_{AB}} \end{pmatrix} = 0 \\
&\mu^2 - 2\Phi\mu + \det(\mathbf{Z}) = 0 \\
&\det(\mathbf{Z}) = -\frac{E_{BA}}{E_{AB}} \\
&\Phi = \frac{1}{2}\left(\frac{E_{BB}}{E_{AB}} - \frac{E_{AA}}{E_{AB}}\right)
\end{aligned} \tag{19}$$

The elements of $\mathbf{E}$ and $\mathbf{Z}$ depend on the energy scattering model. For instance the IEW model for generally asymmetric and inelastic scattering gives [10]

$$\begin{aligned}
&E_{AA} = \left(\left|T^{(+)}\right|^2\left|T^{(-)}\right|^2 - \left|R^{(+)}\right|^2\left|R^{(-)}\right|^2\right)/\left|T^{(-)}\right|^2, \;\; E_{AB} = \left|R^{(-)}\right|^2/\left|T^{(-)}\right|^2 \\
&E_{BA} = -\left|R^{(+)}\right|^2/\left|T^{(-)}\right|^2, \;\; E_{BB} = 1/\left|T^{(-)}\right|^2, \;\; \det(\mathbf{E}) = \left|T^{(+)}\right|^2/\left|T^{(-)}\right|^2
\end{aligned} \tag{20}$$

For asymmetric scatterers, $\det(\mathbf{E}) \neq 1$ and $\det(\mathbf{Z}) \neq 1$. Equations (18) and (19) are symmetrized by the transformations

$$\tilde{\xi} = \frac{1}{\sqrt{\det(\mathbf{E})}}\xi, \;\; \tilde{\Delta} = \frac{1}{\sqrt{\det(\mathbf{E})}}\Delta \tag{21a}$$

$$\tilde{\mu} = \frac{1}{\sqrt{\det(\mathbf{Z})}}\mu, \;\; \tilde{\Phi} = \frac{1}{\sqrt{\det(\mathbf{Z})}}\Phi \tag{21b}$$

[1] The SW phase speed in the definition of flux cancels out of the resultant equations and so is omitted.

BFW, non-BFW and IEW formulae for $\tilde{\Delta}$ and $\tilde{\Omega}$ in terms of $T^{(\pm)}$, $R^{(\pm)}$ and energy loss parameters $\sigma^{(\pm)}$ are given elsewhere [6,10]. Previous papers also pointed out the "mirror" symmetry relationship $\tilde{\xi} \rightleftarrows \tilde{\mu}$ between persistence and reflectivity for structures that differ only by their interchange of $T^{(\pm)}$ and $R^{(\pm)}$ [9, 10]. Another manifestation of the mirror symmetry is that $\mathbf{Z}$ is derived from $\mathbf{E}$ by the interchanges $T^{(\pm)} \rightleftarrows R^{(\pm)}$.

5. Discussion

This paper shows that scattering methods for deriving PSW properties for infinite periodic structures are extended to finite periodic structures using a 2x2 scattering matrix $\mathbf{M}$. Cell independent PSW for an infinite periodic structure are then shown to be the eigenvectors of $\mathbf{M}$. To generate PSW vectors that are eigenvectors requires energy sources with particular properties, such as sources placed at both ends of a finite periodic structure. More generally, PSW vectors are superpositions of the two eigenvectors.

This paper shows that coherent wave propagation is a feature of periodic structures with only symmetric scatterers, whereas to satisfy CoE two or more asymmetric scatterers must modify the SW between them to cancel phase spatial dependence. This is perhaps phase randomisation by multiple reflections between asymmetric scatterers. A more detailed physical model of this effect than what scattering theory delivers is needed. Such a model should also resolve what properties of an asymmetric system that determines which of the two possible solutions apply in the case of symmetric magnitudes of scattering coefficients but asymmetric phase shifts.

REFERENCES

1 Brillouin, L., *Wave Propagation in Periodic Structures*, Second Edition, Dover Publications, Inc., (1953).
2 Ashcroft, N.W. and Mermin, N.D., *Solid State Physics*, Holt, Rinehart and Winston, (1976).
3 Kittel, C., *Introduction to Solid State Physics*, Eighth Edition, John Wiley & Sons Pub., (2004).
4 Cremer, L., Heckl, M. and Ungar, E. E., *Structure Borne Sound*, Second Edition, Springer-Verlag, 405–415, (1973).
5 Fleury, R., Sounas, D. L., Haberman, M. R., Alù, A. Nonreciprocal Acoustics, *Acoustics Today*, **11** (3), 14 – 21, (2015).
6 McMahon, D. Relative performance of different strategies for wave attenuation by periodic structures, *Proceedings of Acoustics 2017*, Perth, Australia, 19-22 November, (2017).
7 McMahon, D. Wave propagation in infinite periodic structures taking into account energy absorption, *Proceedings of Acoustics 2015*, Hunter Valley, Australia, 15-18 November, (2015).
8 McMahon, D. Wave propagation in infinite periodic structures, *Proceedings of the 23rd International Congress on Sound and Vibration* (ICSV23), Athens, Greece, 10-14 July, (2016).
9 McMahon, D. Symmetry of waves in an infinite 1D periodic structure under the interchange of forward and backward scattering coefficients, *Proceedings of Acoustics 2016,* Brisbane, Australia, 9-11 November, (2016).
10 McMahon D. Wave propagation and attenuation in an infinite periodic structure of asymmetric scatterers, *Proceedings of the 24th International Congress on Sound and Vibration* (ICSV24), London, UK. 23-27 July, (2017).