Coherent waves in finite, asymmetric, one dimensional periodic structures

Darryl McMahon

Centre for Marine Science and Technology, Curtin University, Bentley WA 6102, Australia

ABSTRACT

The wave properties for a one dimensional periodic structure are related to the eigenvalues and eigenvectors of two pairs of 2x2 matrices $\mathbf{M}$ and $\mathbf{N}$, $\mathbf{E}$ and $\mathbf{G}$. $\mathbf{M}$ is the scattering matrix of coherent complex waves by a single scatterer while $\mathbf{N}$ is constructed from $\mathbf{M}$ by interchanging eigenvalue and eigenvector parameters. $\mathbf{E}$ is the scattering matrix for wave energy fluxes onto and away from the scatterer, while $\mathbf{G}$ is constructed from $\mathbf{E}$ by interchanging eigenvalue and eigenvector parameters. Equivalently $\mathbf{N}$ and $\mathbf{G}$ are related to $\mathbf{M}$ and $\mathbf{E}$ respectively by interchanging forward and backward scattering parameters. This matrix method allows wave solutions for any size structure, including infinite periodic structures where their "cell independent" vectors are just the eigenvectors of $\mathbf{M}$, $\mathbf{N}$, $\mathbf{E}$ and $\mathbf{G}$. It is shown that the characteristic equations (CE) for the eigenvalues of $\mathbf{E}$ and $\mathbf{G}$ can be derived from the CE for the eigenvalues of $\mathbf{M}$ and $\mathbf{N}$. Further, all the elements of $\mathbf{E}$ and $\mathbf{G}$ can be derived from $\mathbf{M}$ and $\mathbf{N}$ CE parameters. Damped or amplified Bloch-Floquet waves (BFW) are an example of coherent periodic structure waves (PSW) where the difference of backward and forward average phase shifts is $\delta = \pm\pi/2$. More generally scatterers may have internal wave modes giving rise to phase sensitive wave-scatterer energy exchanges causing the difference of backward and forward average phase shifts δ to deviate from $\pm\pi/2$. Phase sensitive energy exchanges allow coherent waves to exist in asymmetric periodic structures, unlike previous papers by the author where asymmetric phase shift differences δ and phase insensitive inelastic scattering alone confined PSW to incoherent energy propagation.

1 INTRODUCTION

Although wave propagation in a periodic structure can have complicated features, such as the passing and stopping wavenumber bands of BFW, they ultimately originate from the properties of a single cell of the structure. The theory is greatly simplified by identifying complex eigenvalues and eigenvectors of a matrix $\hat{\mathbf{M}}$ for one cell, and then constructing the eigenvalues and eigenvectors of $\hat{\mathbf{M}}^m$ for any number *m* of cells, that is, any size periodic sub-structure. The eigenvectors for a periodic sub-structure are also the eigenvectors for a single cell, but the eigenvalues are nonlinear functions of the eigenvalues of a single cell (McMahon 2018). On a larger scale the whole structure may be many sub-structures of different types of cells, and again the eigenvalue/eigenvector technique can be applied since boundary conditions are satisfied by matching superpositions of the eigenvectors of each sub-section at the interfaces.

One can go further and reduce the wave properties of a single cell to the geometry and materials within the cell. For mathematical convenience previous papers only considered 1D structures with a cell consisting of uniform structure material and a point scatterer at one end. More generally this scatterer need not be a point but have some thickness *b*. Then the cell spacing *d* in the point scatterer theory becomes the separation of the front and back surfaces of scatterers in adjacent cells, revising the cell spacing to *d*+*b*. Given the scattering matrix $\mathbf{M}$ for a single scatterer, $\hat{\mathbf{M}}$ is derived by taking into account phase factors $e^{\pm ik_s d}$ from plane wave propagation between scatterers where k_s is the structure material wavenumber.

Scattering is defined by 8 parameters of a complex 2x2 scattering matrix $\mathbf{M}$ transforming any complex wave vector $\begin{pmatrix} A_n & B_n \end{pmatrix}^T$ on one side to $\begin{pmatrix} A_{n+1} & B_{n+1} \end{pmatrix}^T = \mathbf{M}\begin{pmatrix} A_n & B_n \end{pmatrix}^T$ on the other side of the n^{th} scatterer. In this vector notation the "*A*" component is the complex amplitude of a plane wave travelling in the +*x* direction while the "*B*" component travels in the –*x* direction. The 8 parameters are easiest understood in terms of 4 complex scattering coefficients $T^{(\pm)}$ and $R^{(\pm)}$ where $T^{(+)}$ and $R^{(+)}$ are forward transmission and reflection coefficients of the "*A*" component, and $T^{(-)}$ and $R^{(-)}$ apply to the "B" component. From $\mathbf{M}$ written in terms of $T^{(\pm)}$ and $R^{(\pm)}$ another scattering matrix $\mathbf{N}$ can be defined by simply interchanging forward and backward scattering coefficients. Indeed, two periodic structures can be envisaged where their PSW properties are related by $T^{(\pm)} \rightleftarrows R^{(\pm)}$ symmetry (McMahon 2016, 2017a). These two structures were denoted "mirror" structures however it is inappropriate to name $\mathbf{N}$ the "mirror" of $\mathbf{M}$ since mirror matrices are already defined in mathematics differently.

Eight independent parameters in $\mathbf{M}$ and $\mathbf{N}$ gives the widest possible mathematical scope for 1D scatterer properties. Structures that are compound linear and/or nonlinear materials or metamaterials (Haberman and Norris 2016) can give rise to nonreciprocal wave propagation (Fleury et al 2015). Deriving the properties of compound scatterers themselves can also exploit matrix methods, even for nonlinear materials for which eigenvalue/eigenvector techniques and software are available (Betcke et al, 2011).

An important consideration for wave propagation is the relation between complex waves and the wave energy flux. Both complex waves and wave energy fluxes can be treated by the abovementioned eigenvalue/eigenvector techniques. From the eigenvalues and eigenvectors of $\mathbf{M}$ and $\mathbf{N}$, two $T^{(\pm)} \rightleftarrows R^{(\pm)}$ symmetry related 2x2 matrices $\mathbf{E}$ and $\mathbf{G}$ for energy flux scattering can be derived. Some types of energy flux problems can then be solved using the eigenvalues and eigenvectors of $\mathbf{E}$ and $\mathbf{G}$. Except when wave-scatterer energy exchanges are phase insensitive, such as for BFW, the same $\mathbf{E}$ and $\mathbf{G}$ do not apply for any superposition of the eigenvectors of $\mathbf{M}$ and $\mathbf{N}$. Whereas the linearity of $\mathbf{M}$ and $\mathbf{N}$ means they transform any wave vector superposition of eigenvectors, the nonlinearity of energy fluxes means superpositions of $\mathbf{M}$ or $\mathbf{N}$ eigenvectors do not give a linear superpositions of the corresponding eigenvectors of $\mathbf{E}$ or $\mathbf{G}$. To derive the correct 2x2 matrices $\bar{\mathbf{E}}$ or $\bar{\mathbf{G}}$ for a superposition of $\mathbf{M}$ or $\mathbf{N}$ eigenvectors, the superposition is redefined to be an eigenvector for transformed matrices $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ from which $\bar{\mathbf{E}}$ and $\bar{\mathbf{G}}$ are then derived. Similarly, whereas $\hat{\mathbf{M}}^m$ and $\hat{\mathbf{N}}^m$ give complex wave scattering for m cells of a periodic sub-structure, $\hat{\mathbf{E}}^m$ and $\hat{\mathbf{G}}^m$ are not the correct energy flux scattering matrices but rather $\hat{\mathbf{E}}^{(m)}$ and $\hat{\mathbf{G}}^{(m)}$ must be constructed from $\hat{\mathbf{M}}^m$ and $\hat{\mathbf{N}}^m$.

Section 2 summarises the properties of $\mathbf{M}$ and $\mathbf{N}$, and relationships between their eigenvalues and eigenvectors. To help interpret the physical meanings of the eight parameters of $T^{(\pm)}$ and $R^{(\pm)}$, scatterer asymmetry is split into incoherent damping/amplification parts and coherent (phase sensitive) parts. The transformation of $\mathbf{M}$ and $\mathbf{N}$ for isolated scatterers to $\hat{\mathbf{M}}$ and $\hat{\mathbf{N}}$ for scatterers embedded in a periodic structure are derived. Section 3 derives $\mathbf{E}$ and $\mathbf{G}$ corresponding to $\mathbf{M}$ and $\mathbf{N}$ using a property that any 2x2 matrix defines another matrix from which all their elements can be rewritten in terms of their CE parameters. Section 4 derives expressions for wave-scatterer energy exchanges in terms of $\mathbf{E}$ and $\mathbf{G}$ elements. Energy exchanges generally involve interfering incoherent and coherent (phase sensitive) damping/amplification parts. Previous papers only considered incoherent damping energy exchanges which, except for BFW, restrict PSW to incoherent energy waves (IEW) and non-BFW (McMahon 2017a, 2017b). Extending past work to allow possible amplification of waves by scatterers, as exemplified for instance by the thermoacoustic effect (Swift 1988), Sect. 4 derives a formula for phase sensitive wave-scatterer energy exchange. Depending on scattering phase shifts, waves can be either damped or amplified by the latter.

2 Eigenvalues and eigenvectors for $\mathbf{M}$ and $\mathbf{N}$

A useful concept for wave propagation in a 1D periodic structure is the symmetry between forward and backward scattering (McMahon 2016, 2017a, 2017b). This can be understood mathematically from the general fact that any 2x2 scattering matrix $\mathbf{M}$ defines a related scattering matrix $\mathbf{N}$ for which the roles of forward and backward scattering are interchanged. Denoting the two eigenvectors of $\mathbf{M}$, selecting an arbitrary scale factor, as $\begin{pmatrix}1 & \rho^{(\pm)}\end{pmatrix}^T$ with eigenvalues $\gamma^{(\pm)}$ for forward scattering, $\mathbf{N}$ has eigenvectors $\begin{pmatrix}1 & \gamma^{(\pm)}\end{pmatrix}^T$ with eigenvalues $\rho^{(\pm)}$ for backward reflection. Hence $\mathbf{M}$ and $\mathbf{N}$ are related by interchanged eigenvalues and eigenvector components.

The matrix elements of $\mathbf{N}$ are then easily constructed from $\mathbf{M}$. As shown in Sect. 3, 2x2 energy flux matrices $\mathbf{E}$ and $\mathbf{G}$ can be derived where their 4 eigenvalues are the "persistence" $\xi^{(\pm)} = \left|\gamma^{(\pm)}\right|^2$ and reflectivity $\mu^{(\pm)} = \left|\rho^{(\pm)}\right|^2$. The eigenvalue/eigenvector relations for $\mathbf{M}$ are

$$\begin{pmatrix} M_{AA} & M_{AB} \\ M_{BA} & M_{BB} \end{pmatrix} \begin{pmatrix} 1 \\ \rho^{(\pm)} \end{pmatrix} = \gamma^{(\pm)} \begin{pmatrix} 1 \\ \rho^{(\pm)} \end{pmatrix} \tag{1a}$$

with CE and solutions

$$\begin{aligned} &\gamma^2 - 2\Gamma\gamma + \det(\mathbf{M}) = 0,\ \Gamma = \frac{1}{2}\left(M_{AA} + M_{BB}\right) \\ &\gamma^{(\pm)} = \Gamma - (\pm)\sqrt{\Gamma^2 - \det(\mathbf{M})},\ \ \gamma^{(+)}\gamma^{(-)} = \det(\mathbf{M}) \end{aligned} \tag{1b}$$

For $\mathbf{N}$ any vector is written as $\left(X_n \quad Y_n\right)^T$ that is transformed by $\left(X_{n+1} \quad Y_{n+1}\right)^T = \mathbf{N}\left(X_n \quad Y_n\right)^T$. Then the eigenvalue/eigenvector relations for $\mathbf{N}$ are

$$\begin{pmatrix} N_{XX} & N_{XY} \\ N_{YX} & N_{YY} \end{pmatrix} \begin{pmatrix} 1 \\ \gamma^{(\pm)} \end{pmatrix} = \rho^{(\pm)} \begin{pmatrix} 1 \\ \gamma^{(\pm)} \end{pmatrix} \tag{2a}$$

with CE and solutions

$$\begin{aligned} &\rho^2 - 2\Omega\rho + \det(\mathbf{N}) = 0,\ \Omega = \frac{1}{2}\left(N_{XX} + N_{YY}\right) \\ &\rho^{(\pm)} = \Omega - (\pm)\sqrt{\Omega^2 - \det(\mathbf{N})},\ \ \rho^{(+)}\rho^{(-)} = \det(\mathbf{N}) \end{aligned} \tag{2b}$$

The elements of $\mathbf{N}$ in terms of $\mathbf{M}$ are

$$\begin{pmatrix} N_{XX} & N_{XY} \\ N_{YX} & N_{YY} \end{pmatrix} = \frac{1}{M_{AB}} \begin{pmatrix} -M_{AA} & 1 \\ -\det(\mathbf{M}) & M_{BB} \end{pmatrix} \tag{3}$$

Vector components X and Y are related to A and B only by ratios such as for eigenvectors at the n^{th} scatterer

$$\begin{aligned} &A_{n+1} / A_n = Y_n / X_n = \gamma \\ &X_{n+1} / X_n = B_n / A_n = \rho \end{aligned} \tag{4}$$

2.1 Forward and backward scattering, damping/amplification and phase shift parameters

The elements of $\mathbf{M}$ and $\mathbf{N}$ are related to $T^{(\pm)}$ and $R^{(\pm)}$ similar to the scattering equations of McMahon 2017a, 2018 except the latter include the phase factors from plane waves between scatterers. Then

$$\mathbf{M} = \frac{1}{T^{(-)}} \begin{pmatrix} T^{(-)}T^{(+)} - R^{(-)}R^{(+)} & R^{(-)} \\ -R^{(+)} & 1 \end{pmatrix},\ \det(\mathbf{M}) = \frac{T^{(+)}}{T^{(-)}} \tag{5a}$$

$$\mathbf{N} = \frac{1}{R^{(-)}} \begin{pmatrix} R^{(-)}R^{(+)} - T^{(-)}T^{(+)} & T^{(-)} \\ -T^{(+)} & 1 \end{pmatrix},\ \det(\mathbf{N}) = \frac{R^{(+)}}{R^{(-)}} \tag{5b}$$

$\mathbf{M}$ and $\mathbf{N}$ can then be reduced further to facilitate interpretation in terms of physical processes from the following definitions

$$T^{(\pm)} = e^{i\phi^{(\pm)}}\sigma^{(\pm)}\left|T_0^{(\pm)}\right|,\ R^{(\pm)} = e^{i\chi^{(\pm)}}\sigma^{(\pm)}\left|R_0^{(\pm)}\right|,\ \left|T_0^{(\pm)}\right|^2 + \left|R_0^{(\pm)}\right|^2 = 1,$$
$$\bar{\chi} = \frac{1}{2}\left(\chi^{(+)} + \chi^{(-)}\right),\ \bar{\phi} = \frac{1}{2}\left(\phi^{(+)} + \phi^{(-)}\right),\ \delta = \bar{\chi} - \bar{\phi} \tag{6}$$

making up the 8 independent parameters $\left|T_0^{(\pm)}\right|,\ \sigma^{(\pm)}, \phi^{(\pm)}, \chi^{(\pm)}$. Here $\sigma^{(\pm)}$ define incoherent wave-scatterer energy exchanges $0 \le \sigma^{(\pm)} < 1$ for damping, $\sigma^{(\pm)} > 1$ for amplification. $\phi^{(\pm)}$ are forward scattering phase shifts, and $\chi^{(\pm)}$ are backward scattering phase shifts. Four types of scattering asymmetry can be defined: 1. direction dependent incoherent energy exchanges $\sigma^{(+)} \ne \sigma^{(-)}$, 2. asymmetric forward scattering phase shifts $\phi^{(+)} \ne \phi^{(-)}$, 3. asymmetric backward scattering phase shifts $\chi^{(+)} \ne \chi^{(-)}$ and 4. asymmetric scattering amplitudes $\left|T_0^{(+)}\right| \ne \left|T_0^{(-)}\right|$. Asymmetric phase shifts are not unusual, for instance at the boundary of two refractive media $\chi^{(+)}$ and $\chi^{(-)}$ differ by $\pm\pi$.

2.2 Transformation of isolated scatterer matrices to scatterers embedded in a periodic structure

The vector $\left(A_n \quad B_n\right)^T$ is defined at the front surface of the n^{th} scatterer, and $\left(A_{n+1} \quad B_{n+1}\right)^T$ is the vector at the back surface of the n^{th} scatterer where $\left(A_{n+1} \quad B_{n+1}\right)^T = \mathbf{M}\left(A_n \quad B_n\right)^T$. For a scatterer embedded in a cell, the cell vectors are $\mathbf{M}$ vectors modified by phase factors arising from structure plane waves between the front and back surfaces. Whereas the "A" component of a vector $\left(A_n \quad B_n\right)^T$ is defined at the front surface of the n^{th} scatterer, for the n^{th} cell the "A" component $\hat{A}_n$ is defined at the start of the cell at the back of the previous scatterer, a distance d before the n^{th} scatterer and so satisfies $e^{ik_s d}\hat{A}_n = A_n$. Define the n^{th} cell "B" component $\hat{B}_n$ at the scatterer front surface so $\hat{B}_n = B_n$. Then the scatterer vector $\left(A_n \quad B_n\right)^T$ is related to the cell vector $\left(\hat{A}_n \quad \hat{B}_n\right)^T$ by $\left(A_n \quad B_n\right)^T = \mathbf{C}\left(\hat{A}_n \quad \hat{B}_n\right)^T$ where $\mathbf{C}$ is a 2x2 diagonal matrix of phase factors. On the n^{th} scatterer back surface the "A" component A_{n+1} is also the "A" component of the n+1$^{\text{th}}$ cell hence $\hat{A}_{n+1} = A_{n+1}$. The "B" component B_{n+1} is the plane structure wave propagated back from the n+1$^{\text{th}}$ scatterer so that $e^{ik_s d}\hat{B}_{n+1} = B_{n+1}$. Then $\left(A_{n+1} \quad B_{n+1}\right)^T = \mathbf{D}\left(\hat{A}_{n+1} \quad \hat{B}_{n+1}\right)^T$ where $\mathbf{D}$ is a 2x2 diagonal matrix of phase factors. The scattering equation for cell vectors $\left(\hat{A}_{n+1} \quad \hat{B}_{n+1}\right)^T = \hat{\mathbf{M}}\left(\hat{A}_n \quad \hat{B}_n\right)^T$ is derived from the above equation for isolated scatterer vectors and determines that $\hat{\mathbf{M}} = \mathbf{D}^{-1}\mathbf{M}\mathbf{C}$. Equations (5a,b) extended to cell vectors are transformed to

$$\hat{\mathbf{M}} = \frac{1}{\hat{T}^{(-)}}\begin{pmatrix} \hat{T}^{(-)}\hat{T}^{(+)} - \hat{R}^{(-)}\hat{R}^{(+)} & \hat{R}^{(-)} \\ -\hat{R}^{(+)} & 1 \end{pmatrix},\ \det\left(\hat{\mathbf{M}}\right) = \frac{\hat{T}^{(+)}}{\hat{T}^{(-)}} \tag{7a}$$

$$\hat{\mathbf{N}} = \frac{1}{\hat{R}^{(-)}}\begin{pmatrix} \hat{R}^{(-)}\hat{R}^{(+)} - \hat{T}^{(-)}\hat{T}^{(+)} & \hat{T}^{(-)} \\ -\hat{T}^{(+)} & 1 \end{pmatrix},\ \det\left(\hat{\mathbf{N}}\right) = \frac{\hat{R}^{(+)}}{\hat{R}^{(-)}} \tag{7b}$$

where $\hat{T}^{(\pm)} = e^{ik_s d} T^{(\pm)}, \ \hat{R}^{(\pm)} = e^{ik_s d} R^{(\pm)}$. A consequence of the phase factors $e^{ik_s d}$ is that eigenvectors of $\hat{\mathbf{M}}$ and $\hat{\mathbf{N}}$ are superpositions of the eigenvectors of $\mathbf{M}$ and $\mathbf{N}$.

3 Energy flux scattering matrices E and G for eigenvectors

2x2 matrices $\mathbf{E}$ and $\mathbf{G}$ for energy flux scattering have eigenvectors $\begin{pmatrix}1 & \mu^{(\pm)}\end{pmatrix}^T$ and $\begin{pmatrix}1 & \xi^{(\pm)}\end{pmatrix}^T$ respectively, with corresponding eigenvalues $\xi^{(\pm)}$ and $\mu^{(\pm)}$. Then analogous to Eqs. (1a,b) and (2a,b),

$$\begin{pmatrix} E_{AA} & E_{AB} \\ E_{BA} & E_{BB} \end{pmatrix} \begin{pmatrix} 1 \\ \mu^{(\pm)} \end{pmatrix} = \xi^{(\pm)} \begin{pmatrix} 1 \\ \mu^{(\pm)} \end{pmatrix} \tag{8a}$$

$$\xi^2 - 2\Delta\xi + \det(\mathbf{E}) = 0, \ \Delta = \frac{1}{2}(E_{AA} + E_{BB}), \quad \det(\mathbf{E}) = E_{AA}E_{BB} - E_{AB}E_{BA} \tag{8b}$$

$$\begin{pmatrix} G_{XX} & G_{XY} \\ G_{YX} & G_{YY} \end{pmatrix} \begin{pmatrix} 1 \\ \xi^{(\pm)} \end{pmatrix} = \mu^{(\pm)} \begin{pmatrix} 1 \\ \xi^{(\pm)} \end{pmatrix} \tag{9a}$$

$$\mu^2 - 2\Phi\mu + \det(\mathbf{G}) = 0, \ \Phi = \frac{1}{2}(G_{XX} + G_{YY}), \ \det(\mathbf{G}) = G_{XX}G_{YY} - G_{XY}G_{YX} \tag{9b}$$

Using $\xi^{(\pm)} = \left|\gamma^{(\pm)}\right|^2$ and $\mu^{(\pm)} = \left|\rho^{(\pm)}\right|^2$, the CEs Eqs.(1b) and (2b) for $\mathbf{M}$ and $\mathbf{N}$ lead to the CEs Eqs.(8b) and (9b) for $\mathbf{E}$ and $\mathbf{G}$ where

$$\Delta = |\Gamma|^2 + \sqrt{\left(\Gamma^2 - \det(\mathbf{M})\right)\left(\Gamma^{*2} - \det\left(\mathbf{M}^*\right)\right)}, \quad \det(\mathbf{E}) = \left|\det(\mathbf{M})\right|^2 \tag{10a}$$

$$\Phi = |\Omega|^2 + \sqrt{\left(\Omega^2 - \det(\mathbf{N})\right)\left(\Omega^{*2} - \det\left(\mathbf{N}^*\right)\right)}, \quad \det(\mathbf{G}) = \left|\det(\mathbf{N})\right|^2 \tag{10b}$$

Hence $\mathbf{M}$ and $\mathbf{N}$ determine the 4 CE parameters Δ, Φ, $\det(\mathbf{E})$ and $\det(\mathbf{G})$ for $\mathbf{E}$ and $\mathbf{G}$. Noting that $\mathbf{G}$ can be rewritten in terms of $\mathbf{E}$ matrix elements as

$$\mathbf{G} = \frac{1}{E_{AB}} \begin{pmatrix} -E_{AA} & 1 \\ -\det(\mathbf{E}) & E_{BB} \end{pmatrix} \tag{11}$$

then from Eqs.(8b), (9b) and (11) the matrix elements of $\mathbf{E}$ and $\mathbf{G}$ are given by

$$E_{AA} = \Delta - (\pm)\Phi\Theta, \ E_{AB} = (\pm)\Theta, \ E_{BA} = -(\pm)\det(\mathbf{G})\Theta, \ E_{BB} = \Delta + (\pm)\Phi\Theta \tag{12a}$$

$$G_{XX} = \Phi - (\pm)\frac{\Delta}{\Theta}, \ G_{XY} = (\pm)\frac{1}{\Theta}, \ G_{YX} = -(\pm)\frac{\det(\mathbf{E})}{\Theta}, \ G_{YY} = \Phi + (\pm)\frac{\Delta}{\Theta} \tag{12b}$$

where $\Theta = \sqrt{\det(\mathbf{E}) - \Delta^2} / \sqrt{\det(\mathbf{G}) - \Phi^2}$. Because there are no constraints on $\mathbf{M}$ and $\mathbf{N}$, $\mathbf{E}$ and $\mathbf{G}$ using Eqs.(10a,b) are also completely general including all phase sensitive scattering effects. The sign ambiguities are resolved by requiring self-consistency depending on wave damping or amplification.

3.1 General expression for flux matrix $\mathbf{E}$ in terms of energy absorption or amplification

Conservation of energy (CoE) constrains the elements of $\mathbf{E}$ and $\mathbf{G}$. Unlike previous papers, CoE here does not only consider incoherent energy damping $0 \le \sigma^{(\pm)} < 1$. Different to BFW where $\delta = \pm\pi/2$, CoE constrains PSW with asymmetric phase shift $\delta \ne \pm\pi/2$ to incoherent IEW and non-BFW (McMahon 2017a, 2017b). Coherent PSW with $\delta \ne \pm\pi/2$ require phase sensitive wave-scatterer energy exchanges, such as interaction of PSW with waves internal to the scatterer material. This can be taken into account by defining $\breve{\sigma}^{(\pm)}$ similar to $\sigma^{(\pm)}$ except for including phase sensitive energy exchanges.

Consider the net energy flux each side of a scatterer for an eigenvector $\begin{pmatrix} 1 & \mu^{(\pm)} \end{pmatrix}^T$ of $\mathbf{E}$. The net flux is the difference of "A" and "B" terms which is $1-\mu^{(\pm)}$ on one side, and from the RHS of Eq. (8a), the net flux is $\xi^{(\pm)} - \xi^{(\pm)}\mu^{(\pm)}$ on the other side. Hence the flux exchanged by the scatterer is their difference $\left(1-\xi^{(\pm)}\right)\left(1-\mu^{(\pm)}\right)$. This exchanged flux consists of two parts, the exchanged flux $1-\breve{\sigma}^{(+)2}$ from the "A" flux incident onto the scatterer, and exchanged flux $\left(1-\breve{\sigma}^{(-)2}\right)\xi^{(\pm)}\mu^{(\pm)}$ from the "B" flux incident onto the scatterer. The interrelation of eigenvalues and energy exchange terms can be derived from CoE $\left(1-\xi^{(\pm)}\right)\left(1-\mu^{(\pm)}\right) = 1-\breve{\sigma}^{(+)2} + \left(1-\breve{\sigma}^{(-)2}\right)\xi^{(\pm)}\mu^{(\pm)}$ and hence

$$\mu^{(\pm)} = \frac{\breve{\sigma}^{(+)2} - \xi^{(\pm)}}{1-\breve{\sigma}^{(-)2}\xi^{(\pm)}}, \quad \xi^{(\pm)} = \frac{\breve{\sigma}^{(+)2} - \mu^{(\pm)}}{1-\breve{\sigma}^{(-)2}\mu^{(\pm)}} \tag{13}$$

Equation (13) generalizes a previous result limited to incoherent energy damping (McMahon 2017a, 2017b). The 2 off-diagonal matrix elements of $\mathbf{E}$ satisfying Eq.(13) can be derived and rewritten in terms of $\breve{\sigma}^{(\pm)}$ where $\mathbf{E}$ becomes

$$\mathbf{E} = \begin{pmatrix} E_{AA} & -\left(1-\breve{\sigma}^{(-)2}E_{BB}\right) \\ \frac{1}{\breve{\sigma}^{(-)2}}\left(E_{AA} - \breve{\sigma}^{(+)2}\right) & E_{BB} \end{pmatrix} \tag{14}$$

The same method leads to $\hat{\mathbf{E}}$ and $\hat{\mathbf{G}}$ for scatterers embedded in a periodic structure.

4 Phase insensitive and phase sensitive inelastic wave scattering

4.1 Incoherent and coherent scatterer energy absorption or amplification for arbitrary wave vectors

To demonstrate that wave-scatterer energy exchanges, from the general scattering matrix $\mathbf{M}$, divide into incoherent and coherent parts, consider vectors $\begin{pmatrix} A_n & B_n \end{pmatrix}^T$ and $\begin{pmatrix} A_{n+1} & B_{n+1} \end{pmatrix}^T$ on opposite sides of the n^{th} scatterer. The waves incident onto the scatterer are A_n and B_{n+1} and the scattered waves are A_{n+1} and B_n. The energy fluxes cannot be calculated for energy vectors $\begin{pmatrix} |A_n|^2 & |B_n|^2 \end{pmatrix}^T$ and $\begin{pmatrix} |A_{n+1}|^2 & |B_{n+1}|^2 \end{pmatrix}^T$ using $\mathbf{E}$ given by Eq.(12a) because it only applies for a single eigenvector whereas vectors $\begin{pmatrix} A_n & B_n \end{pmatrix}^T$ and $\begin{pmatrix} A_{n+1} & B_{n+1} \end{pmatrix}^T$ are generally superpositions of the eigenvectors of $\mathbf{M}$. Instead it is necessary to derive matrix $\bar{\mathbf{M}}$ such that $\begin{pmatrix} A_n & B_n \end{pmatrix}^T$ is an eigenvector (not normalized to $A_n = 1$) and hence using the above theory derive $\bar{\mathbf{E}}$ such that $\begin{pmatrix} |A_n|^2 & |B_n|^2 \end{pmatrix}^T$ is an eigenvector.

Grouping the input waves into a vector $\left(A_n \quad B_{n+1}\right)^T$ and output waves into a vector $\left(A_{n+1} \quad B_n\right)^T$, the scattering equation can be written as $\left(A_{n+1} \quad B_n\right)^T = \mathbf{S}\left(A_n \quad B_{n+1}\right)^T$ where from Eq.(5a) we find

$$\mathbf{S} = \begin{pmatrix} T^{(+)} & R^{(-)} \\ R^{(+)} & T^{(-)} \end{pmatrix} \tag{15}$$

Hence the output scattered waves are $A_{n+1} = T^{(+)}A_n + R^{(-)}B_{n+1},\ \ B_n = R^{(+)}A_n + T^{(-)}B_{n+1}$ and the net output flux is $\left|A_{n+1}\right|^2 + \left|B_n\right|^2$. The net exchanged energy flux (positive if absorbed, negative if emitted) is the difference of input and output fluxes $F_{ex} = \left|A_n\right|^2 + \left|B_{n+1}\right|^2 - \left(\left|A_{n+1}\right|^2 + \left|B_n\right|^2\right)$ which is the sum of the incoherent (phase insensitive) and coherent (phase sensitive) parts given by

$$F_{ex-incoh} = \left(1 - \left(\left|T^{(+)}\right|^2 + \left|R^{(+)}\right|^2\right)\right)\left|A_n\right|^2 + \left(1 - \left(\left|T^{(-)}\right|^2 + \left|R^{(-)}\right|^2\right)\right)\left|B_{n+1}\right|^2 \tag{16a}$$

$$F_{ex-coh} = -\left(\left(T^{(+)}R^{(-)*} + T^{(-)*}R^{(+)}\right)A_n B_{n+1}{}^* + \left(T^{(+)*}R^{(-)} + T^{(-)}R^{(+)*}\right)A_n{}^* B_{n+1}\right) \tag{16b}$$

Denoting the complex input waves at the scatterer surfaces as $A_n = \left|A_n\right|e^{i\alpha_n}$ and $B_{n+1} = \left|B_{n+1}\right|e^{i\beta_{n+1}}$, obtaining $T^{(\pm)}$ and $R^{(\pm)}$ from Eq.(6) but simplified by $\left|T_0^{(+)}\right| = \left|T_0^{(-)}\right| = \left|T_0\right|,\ \left|R_0^{(+)}\right| = \left|R_0^{(-)}\right| = \left|R_0\right|$, then the RHS of Eqs.(16a,b) give

$$F_{ex-incoh} = \left(1 - \sigma^{(+)2}\right)\left|A_n\right|^2 + \left(1 - \sigma^{(-)2}\right)\left|B_{n+1}\right|^2 \tag{17a}$$

$$F_{ex-coh} = -4\sigma^{(+)}\sigma^{(-)}\left|T_0\right|\left|R_0\right|\cos(\delta)\cos\left(\alpha_n - \beta_{n+1} + \left(\chi^{(+)} - \chi^{(-)}\right)/2 + \left(\phi^{(+)} - \phi^{(-)}\right)/2\right)\left|A_n\right|\left|B_{n+1}\right| \tag{17b}$$

Equation (17a) shows independent energy exchanges for the two input wave fluxes while Eq.(17b) shows that coherent energy exchange requires two simultaneous, opposite travelling input waves. The coherent energy exchange effect was observed previously (McMahon 2018) in the case of elastic scattering by m=2 point scatterers for a single input A_1. The front scatterer closest to the source is impacted by both the incident wave and the wave back-scattered by the second scatterer giving a phase difference $\alpha_1 - \beta_2$ that depends on $k_s d$ and hence a wavenumber dependent coherent energy exchange effect consistent with Eq.(17b). Note that for coherent BFW where $\delta = \pm\pi/2$, phase sensitive energy absorption or amplification is zero. For $\delta \neq \pm\pi/2$, setting $F_{ex-coh} = 0$ constrains energy fluxes to incoherent waves such as IEW and non-BFW requiring scattering that decoheres waves.

4.2 Phase insensitive and phase sensitive wave-scatterer energy exchanges for eigenvectors

Combining Eq. (14) with Eq.(12a), it is possible to write $\breve{\sigma}^{(\pm)}$ in terms of $\mathbf{E}$ and $\mathbf{G}$ CE parameters:

$$\breve{\sigma}^{(+)2} = \frac{1}{\Delta^2 \det(\mathbf{G}) - \Phi^2 \det(\mathbf{E})}\begin{pmatrix} \Delta^2\Phi \det(\mathbf{G}) - \Phi^2\Delta \det(\mathbf{E}) + (\Delta - \Phi)\det(\mathbf{G})\det(\mathbf{E}) \\ +(\pm)\left(\Delta \det(\mathbf{G}) - \Phi \det(\mathbf{E})\right)\sqrt{\left(\det(\mathbf{E}) - \Delta^2\right)\left(\det(\mathbf{G}) - \Phi^2\right)} \end{pmatrix}$$
$$\breve{\sigma}^{(-)2} = \frac{1}{\Delta^2 \det(\mathbf{G}) - \Phi^2 \det(\mathbf{E})}\begin{pmatrix} \Delta \det(\mathbf{G}) - \Phi \det(\mathbf{E}) + \Delta\Phi(\Delta - \Phi) \\ +(\pm)(\Delta - \Phi)\sqrt{\left(\det(\mathbf{E}) - \Delta^2\right)\left(\det(\mathbf{G}) - \Phi^2\right)} \end{pmatrix} \tag{18}$$

Using symmetrised parameters $\tilde{\Delta}=\Delta/\sqrt{\det(\mathbf{E})}$, $\tilde{\Phi}=\Phi/\sqrt{\det(\mathbf{G})}$, we introduce Ψ and $\Lambda_{(\pm)}$ where

$$\Psi=\left(\tilde{\Delta}\tilde{\Phi}+1\right)/\left(\tilde{\Delta}+\tilde{\Phi}\right) \tag{19a}$$

$$\Lambda_{(\pm)}=\Psi+(\pm)\sqrt{\Psi^2-1} \tag{19b}$$

Equations (18) become

$$\begin{aligned}\breve{\sigma}^{(+)2}&=\frac{1}{\tilde{\Delta}-\tilde{\Phi}}\left(-\left(\sqrt{\det(\mathbf{G})}-\sqrt{\det(\mathbf{E})}\right)+\left(\tilde{\Delta}\sqrt{\det(\mathbf{G})}-\tilde{\Phi}\sqrt{\det(\mathbf{E})}\right)\Lambda_{(\pm)}\right)\\ \breve{\sigma}^{(-)2}&=\frac{1}{\tilde{\Delta}-\tilde{\Phi}}\left(-\left(\frac{1}{\sqrt{\det(\mathbf{G})}}-\frac{1}{\sqrt{\det(\mathbf{E})}}\right)+\left(\tilde{\Delta}\frac{1}{\sqrt{\det(\mathbf{G})}}-\tilde{\Phi}\frac{1}{\sqrt{\det(\mathbf{E})}}\right)\Lambda_{(\pm)}\right)\end{aligned} \tag{20}$$

From Eqs.(5a,b), (6) and (10a,b), $\sqrt{\det(\mathbf{E})}=\sqrt{\det(\mathbf{G})}$ is a symmetry property for forward and backward scattering when $\left|T_0^{(+)}\right|=\left|T_0^{(-)}\right|=\left|T_0\right|$, $\left|R_0^{(+)}\right|=\left|R_0^{(-)}\right|=\left|R_0\right|$, then Eqs.(20) simplify to

$$\breve{\sigma}^{(+)2}=\sqrt{\det(\mathbf{E})}\Lambda_{(\pm)},\quad \breve{\sigma}^{(-)2}=\Lambda_{(\pm)}/\sqrt{\det(\mathbf{E})} \tag{21}$$

Also from Eqs.(5a,b), (6) and (10a,b), $\det(\mathbf{E})=\left(\sigma^{(+)}/\sigma^{(-)}\right)^2$ so that Eqs.(21) can be combined to

$$\frac{\breve{\sigma}^{(\pm)2}}{\sigma^{(\pm)2}}=\frac{\Lambda_{(+)}}{\tilde{\sigma}^2},\ \tilde{\sigma}^2>1,\quad \frac{\breve{\sigma}^{(\pm)2}}{\sigma^{(\pm)2}}=\frac{\Lambda_{(-)}}{\tilde{\sigma}^2},\ \tilde{\sigma}^2\le 1 \tag{22}$$

where $\tilde{\sigma}^2=\sigma^{(+)}\sigma^{(-)}$. For $F_{ex-coh}=0$ by $\delta=\pm\pi/2$ from Eq.(17b), evaluation of Ψ gives $\Psi=\Psi_0$ where

$$\Psi_0=\frac{1}{2}\left(\tilde{\sigma}^2+\frac{1}{\tilde{\sigma}^2}\right)\ge 1 \tag{23}$$

$F_{ex-coh}=0$ is also the condition $\Lambda_{(\pm)}=\tilde{\sigma}^2$ which has two self-consistent solutions $\tilde{\sigma}^2=\Psi_0+\sqrt{\Psi_0^{\,2}-1}\ge 1$ and $\tilde{\sigma}^2=\Psi_0-\sqrt{\Psi_0^{\,2}-1}\le 1$, hence giving the $\tilde{\sigma}^2$ conditions in Eq.(22). Ψ_0 applies to all wave types with incoherent wave-scatterer energy exchanges alone, including BFW, incoherent IEW and non-BFW previously identified for infinite periodic structures.

Equation (17b) shows that coherent wave-scatterer energy exchanges require two simultaneous opposite travelling waves and an average backward-forward phase shift difference δ such that $\cos(\delta)\neq 0$. From Eq.(19a) we find with coherent energy exchanges included, Ψ is given by

$$\Psi=\Psi_0+\mathrm{X}\cos^2(\delta) \tag{24}$$

where X is a complicated function (not shown) of scattering parameters in Eq.(6) including δ.

4.3 Examples of phase insensitive and phase sensitive inelastic scatterering for eigenvectors

Figure 1a plots Ψ versus δ/π for an asymmetric scattering model where $\left|T_0\right|=\left|R_0\right|=1/\sqrt{2}$, $\bar{\phi}=0$, $\sigma^{(+)}=1$, $\sigma^{(-)}=0.8,\ 0.6,\ 0.5,\ 0.4,\ 0.3$ while Fig.1b plots Ψ versus δ/π for the same parameters except that

$\sigma^{(-)} = 1/0.8,\ 1/0.6,\ 1/0.5,\ 1/0.4,\ 1/0.3$. The dashed curves are Ψ_0 and show that, for both cases $\tilde{\sigma}^2 \le 1$ and $\tilde{\sigma}^2 > 1$, $1 \le \Psi \le \Psi_0$ and $\Psi = \Psi_0$ at $\delta/\pi = \pm 1/2$.

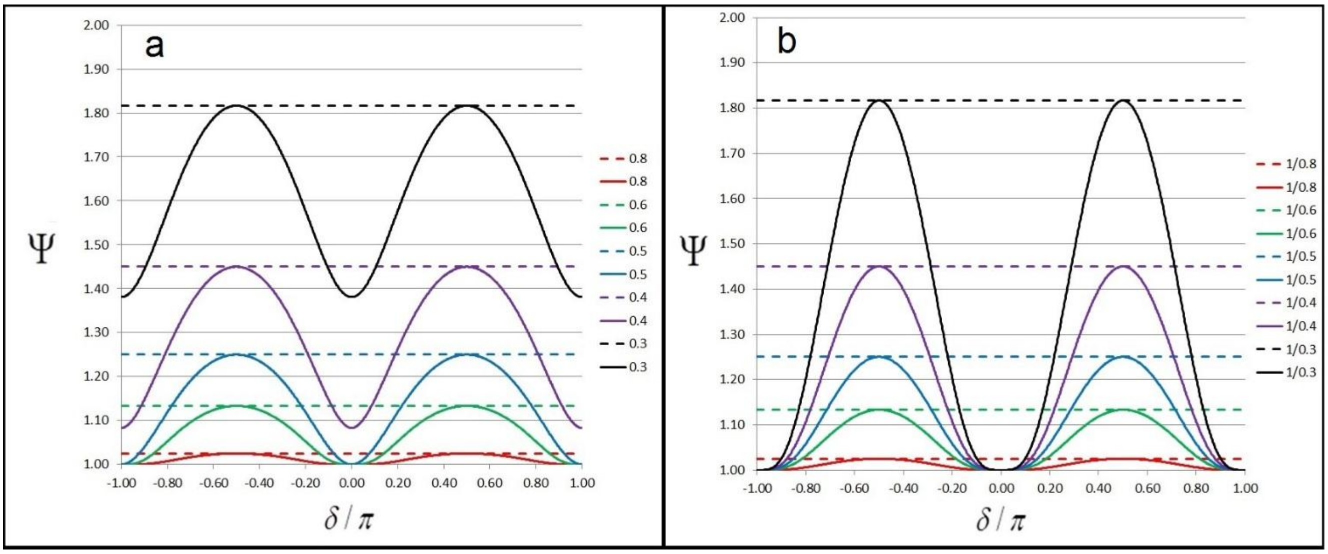

Figure 1: Ψ versus δ/π for $|T_0| = |R_0| = 1/\sqrt{2}$, $\bar{\phi} = 0$, $\sigma^{(+)} = 1$. a. $\tilde{\sigma}^2 \le 1$ for $\sigma^{(-)}$ tabulated at the RHS. b. $\tilde{\sigma}^2 > 1$ for $\sigma^{(-)}$ tabulated at the RHS.

For the same parameters as Figs. 1a,b, Fig. 2a plots $\Lambda_{(-)}/\tilde{\sigma}^2$ for $\tilde{\sigma}^2 \le 1$ and Fig. 2b plots $\Lambda_{(+)}/\tilde{\sigma}^2$ for $\tilde{\sigma}^2 > 1$. Figure 2a shows that for $\tilde{\sigma}^2 \le 1$, $\breve{\sigma}^{(\pm)2}/\sigma^{(\pm)2}$ is increased by coherent wave-scatterer energy exchanges meaning in this case that $\breve{\sigma}^{(+)2} > \sigma^{(+)2} = 1$ is more amplification of the "A" component and $\breve{\sigma}^{(-)2} > \sigma^{(-)2} < 1$ is less damping for the "B" component of the eigenvector. For $\tilde{\sigma}^2 > 1$, Fig. 2b shows that $\breve{\sigma}^{(\pm)2}/\sigma^{(\pm)2}$ is decreased so that $\breve{\sigma}^{(+)2} < \sigma^{(+)2} = 1$ damps the "A" component and $1 < \breve{\sigma}^{(-)2} < \sigma^{(-)2} > 1$ reduces the "B" component but there is still overall amplification.

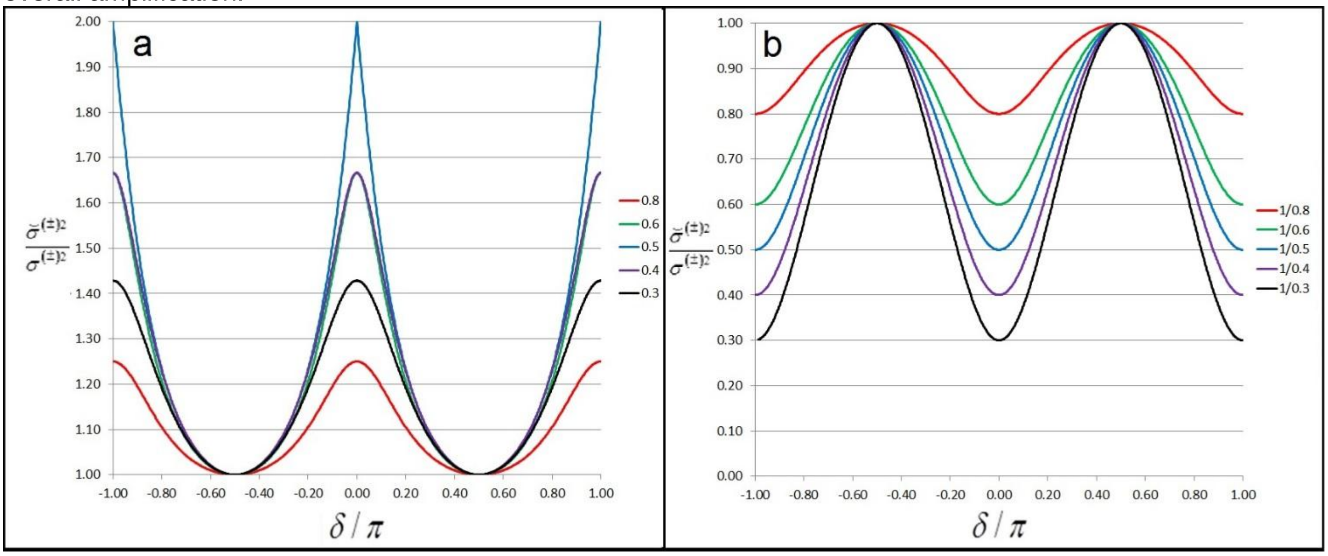

Figure 2: $\breve{\sigma}^{(\pm)2}/\sigma^{(\pm)2}$ versus δ/π for $|T_0| = |R_0| = 1/\sqrt{2}$, $\bar{\phi} = 0$, $\sigma^{(+)} = 1$. a. $\Lambda_{(-)}/\tilde{\sigma}^2$ for $\tilde{\sigma}^2 \le 1$ and $\sigma^{(-)}$ tabulated at the RHS. b. $\Lambda_{(+)}/\tilde{\sigma}^2$ for $\tilde{\sigma}^2 > 1$ and $\sigma^{(-)}$ tabulated at the RHS.

5 SUMMARY

This paper extends previous work on wave propagation and attenuation in 1D periodic structures by including the possibility that the structure material, modelled as scatterers, can absorb or emit energy that damps or amplifies the wave depending on phase. This phenomenon exists for instance in the thermoacoustic effect (Chen et al 2016). Phase sensitive wave-scatterer energy exchanges are associated with scattering phase shift asymmetry $\delta \neq \pm\pi/2$, unlike BFW where inelastic scattering only entails incoherent (phase insensitive) energy exchanges. BFW however still allow energy damping or amplification that depends on propagation direction.

Developing the theory made use of the symmetry between forward and backward scattering, embodied in the two 2x2 matrices $\mathbf{M}$ and $\mathbf{N}$ for an isolated scatterer, and their transformation to $\hat{\mathbf{M}}$ and $\hat{\mathbf{N}}$ for a scatterer embedded in a periodic structure. This matrix formulation is easily applicable to a periodic structure of any size. From the CE for $\mathbf{M}$ and $\mathbf{N}$, two 2x2 energy flux matrices $\mathbf{E}$ and $\mathbf{G}$ were derived from which the energy flux exchanges between waves and scatterers were obtained for eigenvectors. Using the fact that the same eigenvectors of $\hat{\mathbf{M}}$ and $\hat{\mathbf{N}}$ are also the eigenvectors of $\hat{\mathbf{M}}^m$ and $\hat{\mathbf{N}}^m$ for any m scatterers, the CE for $\hat{\mathbf{M}}^m$ and $\hat{\mathbf{N}}^m$ **is** easily derived from which $\hat{\mathbf{E}}^{(m)}$ and $\hat{\mathbf{G}}^{(m)}$ can be obtained. Hence wave-scatterer energy exchanges is obtained for any size periodic structure. Because of the nonlinearity of $\hat{\mathbf{E}}$ and $\hat{\mathbf{G}}$, in general $\hat{\mathbf{E}}^{(m)} \neq \hat{\mathbf{E}}^m$ and $\hat{\mathbf{G}}^{(m)} \neq \hat{\mathbf{G}}^m$ in cases of phase sensitive wave-scatterer energy exchanges.

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[1] Erratum. The correct vertical axis label for Fig. 4 is $\delta^{(+)}/\pi + 1/2$ where $\delta^{(+)} = \chi^{(+)} - \varphi^{(+)}$. The label δ/π shown is incorrect because δ/π always alternates with wavenumber from +1/2 to -1/2 and back for refractive media.