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Some insight into the Green function of the channel problem

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SUMMARY

The Green function for wave-body interaction in a channel of rectangular cross section is discussed in this paper. The results from an asymptotic analysis are outlined, which lead a numerical solution and some insight into the singular or resonant behavior of the Green function. Also presented is a preliminary analysis towards developing an efficient approximation of the channel Green function.

INTRODUCTION

Apart from direct engineering applications, the study of wave-body interaction in a channel is very important for the quantification of tank wall effects on offshore hydrodynamic model testing. It also has relevance in modelling classical field problems between parallel planes that arise in other physics such as acoustics, electrostatics and electro-magnetics [1].

In the context of potential theory, [2, 3, 4, 5, 6], among others, have numerically studied the hydrodynamic channel problem based on the Green function method. It is understood that the Green function calculation is the most difficult and critical part in the channel effects modeling no matter it is formulated based on the method of images, the method of eigen-functions, or in the form of closed integrals. It appears that there is still a need in understanding the characteristics of the channel Green function and in the facilitation and acceleration of the Green function calculations.

The work to be presented in this paper is a continuation of the study published in [6]. The channel Green function is formulated based on the image representation. The complete series of open-sea Green function images is regrouped as images in a near field, images in a middle field and the rest of the infinite number of images in the complementary far field. The part of the Green function induced by the near field images is evaluated in an exact manner based on the calculation of the open-sea Green function. The part induced by the middle field images is estimated by evaluating in an exact form the propagating waves, ignoring the evanescent wave effect. And finally, the part of the Green function induced by the far field images is asymptotically expressed by a plane-wave approximation plus a parabolic correction, which have equivalent single integral forms instead of the slowly convergent series representation. It is analytically demonstrated that the channel Green function has a square-root singular behaviour near the channel-resonant frequencies. The analysis shows that the influence of the infinite number of images located far from the observation point makes a significant contribution to the channel wall effects. The asymptotic analysis also provides a possibility to further replace the integral representation of the far field solution by an economized polynomial approximation.

BASIC FORMULATION

The Cartesian coordinate system *o-xyz* is defined in this paper as an 'equilibrium' set of axes with *ox* along the longitudinal direction of the wave channel. The $z = 0$ plane corresponds to the calm water level, and z is positive upwards. The *x-z* plane is coincident with the centre-plane of the tank. We assume an ideal fluid and an irrotational flow; ω is the oscillating frequency of the fluid motion and the time dependence is of the form $e^{-\mathrm{i}\omega t}$. The channel Green function is denoted by $G(\boldsymbol{x},\boldsymbol{x}')$ which represents the spatial part of the velocity potential at a field point $\boldsymbol{x}=(x,y,z)$ in the wave channel due to a pulsating source of unit strength at the point $\boldsymbol{x}'=(x',y',z')$. The channel Green function must satisfy the following governing equations

$$\begin{cases} \nabla^2 G(\boldsymbol{x},\boldsymbol{x}') = -4\pi\,\delta(\boldsymbol{x}-\boldsymbol{x}') & \text{in the fluid} \\ -\dfrac{\omega^2}{g}G + \dfrac{\partial G}{\partial z} = 0 & \text{on } z=0 \\ \dfrac{\partial G}{\partial z} = 0 & \text{on } z=-h \\ \dfrac{\partial G}{\partial y} = 0 & \text{on } y=\pm\dfrac{b}{2} \end{cases} \tag{1}$$

where δ is the Dirac delta function, g the gravitational acceleration; h denotes the water depth, b the width of the wave tank. In addition the Green function must satisfy a radiation condition that states that, at infinity, G is associated only with waves that propagate away from the source.

The solution of (1) can be obtained by considering an infinite number of images of the source at the positions $\boldsymbol{x}'_m=(x',y'_m,z')$ with the y-coordinate y'_m determined by $y'_m=(-1)^m y' + mb$. That is

$$G(\boldsymbol{x},\boldsymbol{x}') = \sum_{m=-\infty}^{\infty} G_m \tag{2}$$

where $G_m = G(\boldsymbol{x}, \boldsymbol{x}'_m)$ is the open-sea Green function satisfying the first three equations in (1), which represents the potential at the field point $\boldsymbol{x}$ due to the m-th image of the source at the point $\boldsymbol{x}'_m$. Obviously, $m = 0$ represents the source itself.

The open-sea Green function G_m is given in [7] in the form of principal-value integral as

$$G_m = \frac{1}{r_m} + \frac{1}{r'_m} + P\int_0^\infty F(\mu) J_0(\mu\rho_m) d\mu + \mathrm{i}\,\Lambda J_0(k_0\rho_m) \tag{3}$$

with $\rho_m = \sqrt{(x-x')^2 + (y-y'_m)^2}$; $r_m = \sqrt{\rho_m^2 + (z-z')^2}$; $r'_m = \sqrt{\rho_m^2 + (z+z'+2h)^2}$; J_0 is the first kind of Bessel function; the other functions are defined as

$$F(\mu) = \frac{(\mu+\nu)\,\mathrm{e}^{-2\mu h} \cosh\mu(z+h)\cosh\mu(z'+h)}{(\mu-\nu) - (\mu+\nu)\,\mathrm{e}^{-2\mu h}} \tag{4}$$

$$\Lambda = \frac{2\pi(k_0^2 - \nu^2)\cosh k_0(z+h)\cosh k_0(z'+h)}{(k_0^2-\nu^2)h + \nu} \tag{5}$$

where $\nu = \omega^2 / g$.

Alternatively, a series expansion of (3) presented by John [8], which is much more efficient when ρ_m is not small, can be written as

$$G_m = \zeta_0(z,z') H_0(k_0\rho_m) + \sum_{n=1}^{\infty} \zeta_n(z,z') K_0(k_n\rho_m) \tag{6}$$

where the first term represents the propagating wave and the summation describes the evanescent wave effect; H_0 and K_0 are respectively the Hankel function of the first kind and the modified Bessel function; k_n are defined as the evanescent wave number, which are real positive solutions of the equation $k_n \tan(k_n h) = -\omega^2 / g$. The z-dependent functions in (6) are defined by

$$\zeta_0(z,z') = \mathrm{i}\,\Lambda \tag{7}$$

$$\zeta_n(z,z') = \frac{8k_n}{2k_n h + \sin(2k_n h)} \cos k_n(z+h)\cos k_n(z'+h) \tag{8}$$

Many efforts have been made to achieve accurate and efficient evaluation of the open-sea Green function based on (3) (e.g. [9]). The following describes a procedure for the more difficult channel problem as formally defined by (2).

ASYMPTOTIC APPROACH

Since the calculation of the open-sea Green function may be gradually simplified as the distance between the field and the image point increases, we may rewrite Eq. (2) as

$$G(\boldsymbol{x}, \boldsymbol{x}') = G^N + G^M + G^F \tag{9}$$

where G^N represents the potential induced by the source and its nearest images that require a complete evaluation; G^M defines the potential induced by a finite series of images in the middle field where the evanescent wave effect as described in (6) can be neglected; G^F represents the potential induced by the remaining infinite number of images in the far field where further approximation may be introduced. G^N and G^M may be expressed as

$$G^N = \sum_{m=-M_0}^{M_0} G_m \tag{10}$$

$$G^M = \zeta_0(z,z') \sum_{m=M_0+1}^{2M_1-1} \left[H_0(k_0\rho_m) + H_0(k_0\rho_{-m})\right] \tag{11}$$

Provided all the remaining images are *far enough* from the field point, G^F can be approximated by a plane-wave approximation plus a parabolic correction [6] as

$$G^F = c\zeta_0(z,z') \sum_{l=1}^{4} \left[\eta_1(Y_l, B) + \left(\frac{1}{2}X^2 - \frac{1}{8}\right)\eta_2(Y_l, B)\right] \tag{12}$$

where $c = \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2/\pi}$, $X = k_0(x - x')$ and

$$\eta_1(Y_l, B) = \frac{\mathrm{e}^{2\mathrm{i}BY_l}}{\sqrt{2\pi B}} \int_0^\infty \frac{t^{-1/2}\,\mathrm{e}^{-Y_l t}}{1 - \mathrm{e}^{-t+\mathrm{i}2B}} dt \tag{13}$$

$$\eta_2(Y_l, B) = \frac{\mathrm{i}\mathrm{e}^{2\mathrm{i}BY_l}}{\sqrt{2\pi B}B} \int_0^\infty \frac{t^{1/2}\,\mathrm{e}^{-Y_l t}}{1 - \mathrm{e}^{-t+\mathrm{i}2B}} dt \tag{14}$$

with $B = k_0 b$, the non-dimensional channel width; Y_l $(l = 1,2,3,4)$ are the non-dimensional transverse distances between the field point and the nearest four images in the far field,

$$\begin{cases} Y_1 = M_1 - (y-y')/(2b) \\ Y_2 = M_1 + (y-y')/(2b) \\ Y_3 = M_1 + 1/2 - (y+y')/(2b) \\ Y_4 = M_1 + 1/2 + (y+y')/(2b) \end{cases}$$

In order to guarantee an absolute accuracy of 10^{-5} for ignoring the evanescent waves in the middle and far fields, the truncation number M_0 is approximately bounded by

$$M_0 \geq 1 + \mathrm{int}(7h/b) \tag{15}$$

The truncation number M_1 determines the distance between the field point and the nearest images in the far field, Y_l, and hence the convergence and accuracy of the asymptotic solution (12). In order to guarantee an absolute accuracy of 10^{-5} for replacing the far field images by the asymptotic solution, we may set up the following criterion

$$M_1 > \max\left\{|X|,\ 13.5B^{-1}(X^4+1.5X^2+0.5625)^{\frac{2}{5}},\right.$$
$$\left. 22.1B^{-\frac{5}{3}}(X^4+1.5X^2+0.5625)^{\frac{2}{3}}\right\}+$$
$$(2b)^{-1}|y-y'| \qquad (16)$$

Since the negligence of the evanescent wave effect is a precondition for the far field approximation, another obvious restriction has to be assigned for M_1, i.e. $2M_1 \geq M_0 + 1$. In the case of holding the equality, all the images are included in either near or far field.

The summation over the infinite number of images in the far field is now expressed in (12) by the Y_l-dependent functions $\eta_1(Y_l, B)$ and $\eta_2(Y_l, B)$ which have single integral representations as given by (13) and (14). The single integral representations are more advantageous than the slowly convergent infinite series due to the fact that the kernels are smooth functions with an exponentially decaying component; the numerical evaluation is then easier and faster convergent.

In is not difficult to see that when $B \to n\pi$, $n = 0, 1, 2, 3, \cdots$, $\eta_1(Y_l, B)$ is divergent. In other words, the plane-wave term in (12) exhibits a very important feature of the channel wall effects, that is, channel resonance occurs when the channel width is an integer multiple of half the wavelength. No matter how large the truncation number M_1 is, the singularity exists in the far field solution. Therefore, it is difficult to model the channel resonance by taking only a finite number of images in the Green function calculation.

By defining $\varepsilon = 2|B - n\pi|$ with $n = \mathrm{int}(B/\pi)$, we may rewrite the integral in (13) as

$$I(Y_l, B) = \int_0^\infty \frac{t^{-1/2}\,\mathrm{e}^{-Y_l t}}{1-\mathrm{e}^{-t+\mathrm{i}\,2B}}\,dt = \int_0^\infty \frac{t^{-1/2}\,\mathrm{e}^{-Y_l t}}{1-\mathrm{e}^{-t+\mathrm{i}\,\varepsilon\,\mathrm{sign}(B-n\pi)}}\,dt \qquad (17)$$

In order to derive a leading-order approximation to the channel resonance, we notice that in this case the main contribution to the integral of $\eta_1(Y_l, B)$ is near the singular point $t = 0$. Therefore, by approximating $\mathrm{e}^{-t+\mathrm{i}\,\varepsilon\,\mathrm{sign}(B-n\pi)}$ by $1 + t - \mathrm{i}\,\varepsilon\,\mathrm{sign}(B - n\pi)$, one has

$$I \approx \frac{\pi}{\sqrt{\varepsilon}}\left[-V_{\frac{3}{2}}(2\varepsilon Y_l, 0) + i\,\mathrm{sign}(B-n\pi)V_{\frac{1}{2}}(2\varepsilon Y_l, 0)\right]$$
$$\text{when } \varepsilon \to 0 \qquad (18)$$

where $V_{\frac{1}{2}}$ and $V_{\frac{3}{2}}$ are the so-called Gillbert's integrals or Gillbert's type of Lommel functions [10],

$$V_{\frac{1}{2}}(\lambda, 0) = \frac{1}{\pi}\int_0^\infty \frac{u^{-1/2} e^{-u\lambda/2}}{1+u^2}\,du \qquad (19)$$

$$V_{\frac{3}{2}}(\lambda, 0) = -\frac{1}{\pi}\int_0^\infty \frac{u^{1/2} e^{-u\lambda/2}}{1+u^2}\,du \qquad (20)$$

Since $\lim_{\lambda\to 0} V_{\frac{1}{2}}(\lambda, 0) = -\lim_{\lambda\to 0} V_{\frac{3}{2}}(\lambda, 0) = \frac{1}{\sqrt{2}}$, (18) may be further approximated by

$$I \approx \frac{\pi}{\sqrt{\varepsilon}} e^{i\frac{\pi}{4}\mathrm{sign}(B-n\pi)} \qquad \text{when } \varepsilon \to 0 \qquad (21)$$

Substitution of (21) into (12) gives

$$G^F \approx \frac{2}{\sqrt{B\varepsilon}}\Lambda(z, z')\{\cos[n\pi(y-y')/b]$$
$$+ (-1)^n \cos[n\pi(y+y')/b]\}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}[1+\mathrm{sign}(B-n\pi)]}$$
$$\text{when } \varepsilon \to 0 \qquad (22)$$

This indicates that the real part of G^F has a one-sided square-root singularity when $B \to n\pi$ from the smaller side, while the imaginary part of G^F has the singularity when $B \to n\pi$ from the greater side. The asymptotic expression (22) is independent of Y_l. It holds also for the infinite propagating wave series that includes all the mirror images. As the evanescent wave modes do not exhibit wave resonance, it may be expected that the channel Green function G contains the same singularity as in G^F.

Given a certain water depth, Figure 1 compares the computational results for the open-sea Green function and for the channel Green function of three different channel widths based on (10) – (12).

TOWARDS A CHEBYSHEV APPROXIMATION

In order to accelerate the computation of the Y_l-dependent integrals in $\eta_1(Y_l, B)$ and $\eta_2(Y_l, B)$, one may think of a multivariate economized polynomial approximation based on Chebyshev expansions [9]. Since $\eta_2(Y_l, B)$ is a smooth and regular function the approximation of it may be achieved relatively easily provided the integral is evaluated accurately. In the following, we propose an idea towards an economized polynomial approximation of the singular function $\eta_1(Y_l, B)$.

By virtue of the asymptotic expression (18), the integral (17) may be rewritten as

$$I = I_1 + \frac{\pi}{\sqrt{\varepsilon}}\left[-V_{\frac{3}{2}}(2\varepsilon Y_l, 0) + i\,\mathrm{sign}(B-n\pi)V_{\frac{1}{2}}(2\varepsilon Y_l, 0)\right] \qquad (23)$$

where

$$I_1 = \int_0^\infty \frac{t^{-1/2}\,\mathrm{e}^{-Y_l t}}{1-\mathrm{e}^{-t+\mathrm{i}\,2B}}\,dt$$
$$-\frac{\pi}{\sqrt{\varepsilon}}\left[-V_{\frac{3}{2}}(2\varepsilon Y_l, 0) + i\,\mathrm{sign}(B-n\pi)V_{\frac{1}{2}}(2\varepsilon Y_l, 0)\right] \qquad (24)$$

or

$$I_1 = \int_0^{\infty} \frac{(t-1-\mathrm{i}2B+\mathrm{e}^{-t+\mathrm{i}2B})}{(1-\mathrm{e}^{-t+\mathrm{i}2B})(t-\mathrm{i}2B)} t^{-1/2}\, \mathrm{e}^{-Y_l t}\, dt \qquad (25)$$

is a smooth and regular function that may be approximated through Chebyshev expansions. In Eq. (23), the Gillbert's type of Lommel functions may as well be approximated through Chebyshev expansions.

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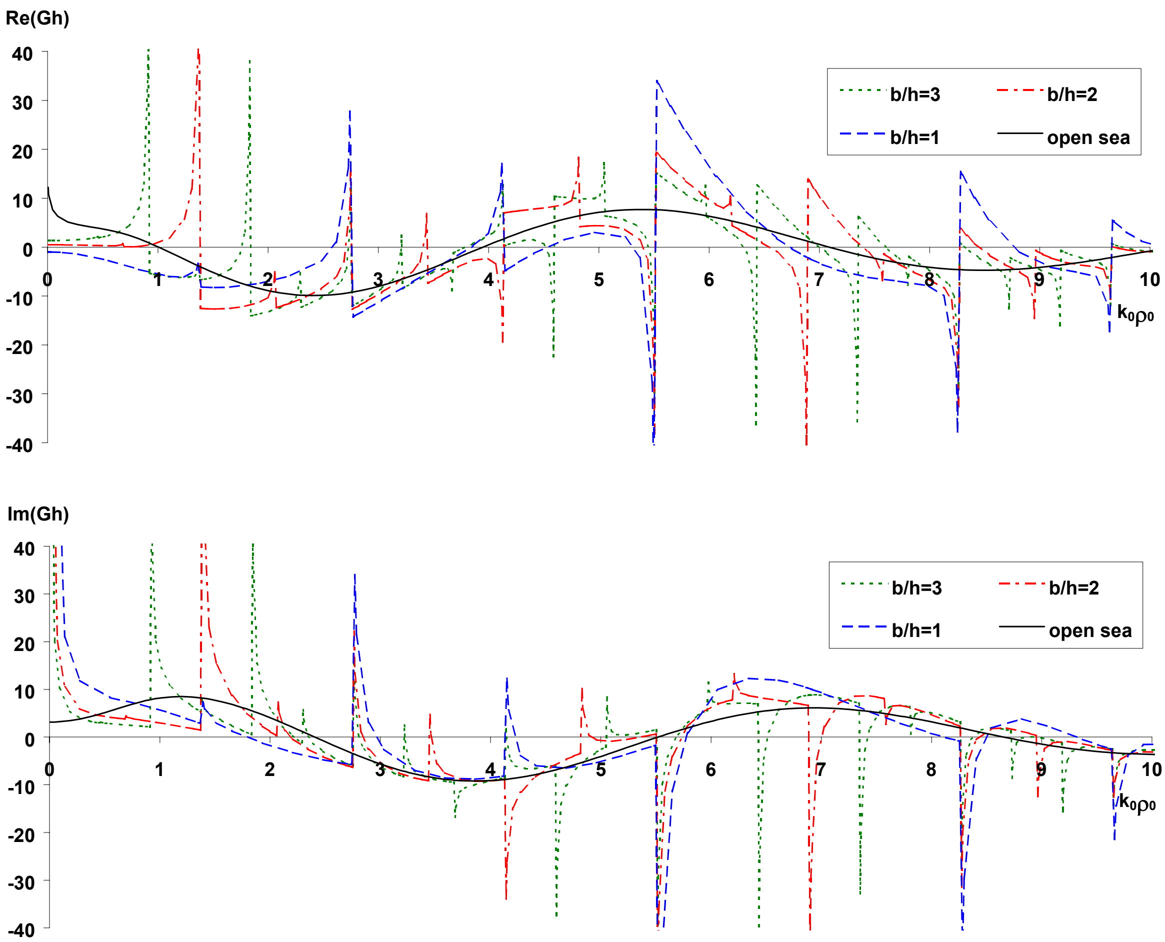

Figure 1. Real part (up) and imaginary part (down) of the channel Green function vs. wave number. $(x, y, z) = (0.4h, 0.02h, -0.05h)$; $(x', y', z') = (0.0, 0.2h, -0.05h)$; $b/h = 1, 2, 3, \infty$