Symmetry of waves in an infinite 1D periodic structure under the interchange of forward and backward scattering coefficients

Darryl McMahon

Centre for Marine Science and Technology, Curtin University, Bentley WA 6102, Australia

ABSTRACT

Wave propagation and attenuation in an infinite periodic structure is determined by the properties of equally spaced scatterers embedded in the structure. These properties are the complex forward and backward scattering coefficients and their dependence on the direction of the structure waves (SW) incident onto the scatterers. The theory for periodic structure waves (PSW) is constrained to be simultaneously consistent with the coupling of SW in adjacent "cells" between the scatterers, and conservation of energy where the scatterers generally absorb energy. Depending on the scattering parameters, PSW can be attenuated Bloch-Floquet waves (BFW), another extended PSW or up to two localized (non-propagating) "modes". The latter two non-BFW modes require by conservation of energy two phase correlated PSW coupled by scattering. It is shown that symmetry relationships must exist between the PSW properties for two different periodic structures that "mirror" each other by the forward scattering coefficient of one being equal to the backward scattering coefficient of the other. Hence deriving PSW results such as the SW reflectivity for one structure also provides corresponding results for its mirror structure.

1. INTRODUCTION

This paper is an extension to recent research on wave phenomena in an infinite 1D periodic structure (McMahon, 2015a,b; 2016). This structure seems very simple, just planar SW propagating between equally spaced identical scatterers. So the complexity arises from an infinite number of boundary conditions created by the scatterers. The most interesting problem to solve is to find all possible solutions for energy transmission and reflection by the structure, which requires developing equations for PSW amplitudes and phases along the structure. This is not a new problem, but has a long history (Brillouin, 1953) with many practical applications. New insights arise however when the properties of the scatterers are made as general as possible, in particular the ability to absorb energy from the SW and no constraints on the scattering phase shifts.

An equation, dubbed the characteristic equation (CE), for a PSW is developed by relating its amplitude and phase in adjacent cells to the scattering coefficients. No explicit consideration of the conservation of energy (CoE) seems necessary to derive the CE for such a simple problem with planar SW. However except for BFW which require highly constrained scattering phase shifts, the CE is not consistent with CoE (McMahon, 2015b). The solution proposed in (McMahon, 2016) is that a single PSW cannot exist for all possible periodic structures, but PSW occur in pairs that are phase correlated. Then it is possible to derive a CE for the PSW pair consistent with CoE. BFW still arise as a special case, but in addition to BFW correlated PSW pairs allow a new type of extended wave that can exist for elastic scattering and up to two localized waves.

This paper expands further on the allowed PSW types and their properties. CoE is shown to limit the maximum number of localized PSW modes to two. This paper explains the extra extended wave solution for periodic structures. Further, CoE reveals a simple connection between the PSW of a structure with the PSW for its mirror structure defined by interchanging the backward reflection coefficient with the forward transmission coefficient.

2. CONSERVATION OF ENERGY CONSTRAINTS ON PERIODIC STRUCTURE WAVES

Consider a 1D periodic structure with equally spaced symmetric structure wave scatterers with spacing d. Scatterers are defined to be symmetric when the magnitudes of the forward transmission coefficient $|T|$ and backward reflection coefficient $|R|$ are the same for both sides the structure wave is incident upon. The forward transmission and backward reflection phase shifts may have the same (SW1) or opposite (SW2) signs for opposite directions of the SW incident onto a scatterer. A single energy source is assumed for $x \rightarrow -\infty$ so overall energy propagation is in the +x direction. In any cell between two scatterers, the phase speed can be omitted from expressions for the time averaged energy flux which for the n^{th} cell we denote as $|A_n|^2$ for SW propagating in the $+x$

direction and $|B_n|^2$ for propagation in the -x direction. The net flux F_i incident onto a scatterer at the edge of the n^{th} and $(n+1)^{th}$ cells is then

$$F_i = |A_n|^2 + |B_{n+1}|^2 \tag{1}$$

and the net flux F_o reflected off a scatterer is

$$F_o = |B_n|^2 + |A_{n+1}|^2 \tag{2}$$

Combining Eqs. (1) and (2) with conservation of energy given by $F_o = \sigma^2 F_i$, where $0 \le 1-\sigma^2 \le 1$ is the proportion of the incident flux absorbed by the scatterer, gives the relationship

$$\frac{|A_{n+1}|^2}{|A_n|^2} = \frac{\sigma^2 - \mu_n}{1-\sigma^2 \mu_{n+1}} \tag{3}$$

where μ_n is the reflectivity of the periodic structure at the n^{th} cell and defined by the ratio of backward and forward energy fluxes

$$\mu_n = \frac{|B_n|^2}{|A_n|^2} \tag{4}$$

From Eqs. (3) and (4) $0 \le \mu_n \le \sigma^2$. Equations (3-4) do not make any assumptions about the location of an energy source or size of the periodic structure and are important general constraints on the solutions to the characteristic equation (CE) for PSW that are compatible with the boundary conditions defined by the scattering coefficients.

For cells a large distance from an energy source, which we consider to be near finite n, the only possible asymptotic property of μ_n is $\lim n \to \infty, \mu_{n+1} \to \mu_n \to \mu$ which from Eq. (3) leads to

$$\lim n \to \infty, \frac{|A_{n+1}|^2}{|A_n|^2} \to |\gamma|^2 = \frac{\sigma^2 - \mu}{1-\sigma^2 \mu} \tag{5}$$

Both $|\gamma|^2$ and μ can be derived from the CE. However, as found in a previous paper (McMahon, 2015b) the CE for a single PSW of an infinite periodic structure is not generally consistent with the CoE, and it is necessary (except for BFW) to introduce two phase correlated PSW that make the CE consistent with the CoE (McMahon, 2016).

2.1 Relationships between pairs of PSW modes with the same energy absorptivity

Whereas Eq.(5) is derived from the energy fluxes toward and away from a single scatterer, another form of CoE can be defined at any cell for energy fluxes reflected, transmitted and absorbed by a periodic structure (McMahon, 2015b). Using Eq.(5) the reflectivity μ, transmissivity η and absorptivity κ of a periodic structure are related to $|\gamma|^2$ by

$$\mu = \frac{\sigma^2 - |\gamma|^2}{1-\sigma^2|\gamma|^2} \tag{6a}$$

$$\eta = (1-\mu)|\gamma|^2 = (1-\sigma^2)\frac{|\gamma|^2(1+|\gamma|^2)}{1-\sigma^2|\gamma|^2} \tag{6b}$$

$$\kappa = 1-\mu-\eta = (1-\sigma^2)\frac{1-|\gamma|^4}{1-\sigma^2|\gamma|^2} \tag{6c}$$

From Eq.(6a) it is clear that $|\gamma|^2$ is constrained by $0 \le |\gamma|^2 \le \sigma^2$. Also Eqs.(6a,b) do not provide any information about μ and η in the case of elastic scattering $\sigma^2 = 1$ for $|\gamma|^2 = 1$ where κ=0. This applies to a BFW passing band for which only the CE can give μ and η. However Eq.(6a) correctly gives μ=1 which applies to a BFW stopping band where $\sigma^2 = 1, |\gamma|^2 < 1$.

From Eqs.(6b,c) we find that for $\sigma^2 < 1$ we have quadratic equations for $|\gamma|^2$ treating η and κ as parameters. Consequently, these CoE considerations reveal the possibility of two different $|\gamma|^2$ and hence two μ, η and κ for the same structure, although both solutions must also satisfy the CE. Thus an infinite periodic structure can have up to two PSW modes, an example of which are the two localized modes found in numerical solutions of the equation for CE merged with the CoE (McMahon, 2016: Eqs.(11a,b) and Fig. 2). Alternatively, for a given κ only one of the two possible $|\gamma|^2$ may satisfy the CE. This is the case of BFW where the CoE provides no new information not already in the CE (McMahon, 2016: Sect. 4.1).

Assume there are two PSW modes with equal κ, denoted as $|\gamma_{(\pm)}|^2$, which from Eq.(6c) are given by

$$|\gamma_{(\pm)}|^2 = \left(\frac{\sigma^2}{2}\right)\left(\frac{\kappa}{1-\sigma^2}\right) + (\pm)\sqrt{\left(\frac{\sigma^2}{2}\right)^2\left(\frac{\kappa}{1-\sigma^2}\right)^2 - \left(\frac{\kappa}{1-\sigma^2} - 1\right)} \tag{7}$$

From the requirements that $|\gamma_{(\pm)}|^2$ are real and satisfy $0 \le |\gamma_{(\pm)}|^2 \le \sigma^2$ we find

$$\kappa_{\min} = 1-\sigma^2 \le \kappa \le \kappa_{\max} = 2\frac{(1-\sigma^2)}{\left(1+\sqrt{1-\sigma^4}\right)} \tag{8}$$

Hence by CoE κ must be between the minimum possible energy absorptivity $1-\sigma^2$ equal to that of an isolated scatterer, and the maximum possible energy absorptivity $\kappa_{\max}$ for a scatterer embedded in an infinite periodic structure. An absorptivity greater than $1-\sigma^2$ arises because the same wave energy can be incident onto a scatterer multiple times owing to reflections and transmissions by the neighboring structure. For $\kappa = \kappa_{\max}$ we find $|\gamma_{(-)}|^2 = |\gamma_{(+)}|^2$ so in this case there is only one PSW mode.

A relationship derived from Eqs.(6a,c) for $\sigma^2 < 1$ is

$$\mu|\gamma|^2 = \frac{\kappa}{(1-\sigma^2)} - 1 \tag{9}$$

Applying Eq.(9) to $|\gamma_{(\pm)}|^2$ and their corresponding reflectivities $\mu_{(\pm)}$, using from Eq.(7)

$$|\gamma_{(+)}|^2|\gamma_{(-)}|^2 = \frac{\kappa}{(1-\sigma^2)} - 1 \tag{10}$$

we find that $\mu_{(\pm)} = |\gamma_{(\mp)}|^2$ for two PSW modes with equal κ and $\sigma^2 < 1$. This also shows that the reflectivities for two PSW modes with equal κ are related by

$$\mu_{(\pm)} = \frac{\sigma^2 - \mu_{(\mp)}}{1-\sigma^2\mu_{(\mp)}} \tag{11}$$

Some of the above results also apply to those periodic structures where only one $|\gamma|^2$ is consistent with both the CoE and CE. Using κ calculated from Eq.(6c), $|\gamma_{(\pm)}|^2$ can still be derived from Eq.(7) where one equals $|\gamma|^2$ (whether it is $|\gamma_{(-)}|^2$ or $|\gamma_{(+)}|^2$ can only be determined from the CE) and the other equals μ. This leads to the suggestion that if both $|\gamma_{(-)}|^2$ and $|\gamma_{(+)}|^2$ cannot be PSW modes for the same structure, then perhaps one is a PSW mode for a different structure that has the same κ and $\sigma^2 < 1$. This indeed is the case for the incoherent wave energy model where the other structure is the mirror structure defined earlier (see Subsect. 3.1). It does not quite apply to BFW however since the mirror structure relationship also requires a phase translation (see Subsect. 3.2).

More generally two localized PSW modes with different values of κ can exist for the same structure. Denoting these as $\kappa^{(n)}, n = 1,2$ then by the obvious extension of Eq.(7) for $\kappa^{(1)} \ne \kappa^{(2)}$ there can be up to four related PSW solutions $\left|\gamma_{(\pm)}^{(n)}\right|^2, n=1,2$, and hence from the above analysis four reflectivities $\mu_{(\mp)}^{(n)} = \left|\gamma_{(\pm)}^{(n)}\right|^2, n=1,2$. However only two of these are for the original structure, so the other two must be for a second but related periodic structure with the same two $\kappa^{(n)}, n=1,2$. Indeed as seen in Subsect. 3.1 the two structures have their reflection and transmission coefficients exchanged (i.e. are mirror structures).

Suppose $\left|\gamma_{(-)}^{(n)}\right|^2, n=1\,\mathrm{or}\,2$ is a PSW mode for the first structure, then $\left|\gamma_{(+)}^{(n)}\right|^2, n=1\,\mathrm{or}\,2$ is a PSW of the second structure where from Eq.(7) it is easily seen that

$$\left|\gamma_{(+)}^{(n)}\right|^2 = \frac{\sigma^2 - \left|\gamma_{(-)}^{(n)}\right|^2}{1-\sigma^2\left|\gamma_{(-)}^{(n)}\right|^2}, n=1,2 \tag{12}$$

This leaves the question which of the two $\left|\gamma_{(\pm)}^{(m)}\right|^2, m \neq n$ belongs to each structure. Using the fact that two PSW modes require $\kappa < \kappa_{\max}$ and hence $\left|\gamma_{(+)}^{(n)}\right|^2 > \left|\gamma_{(-)}^{(n)}\right|^2, n=1,2$, combined with Eq.(12) we find

$$\begin{aligned} &\left|\gamma_{(-)}^{(n)}\right|^2 < \left|\gamma_0\right|^2, n=1,2 \\ &\left|\gamma_{(+)}^{(n)}\right|^2 > \left|\gamma_0\right|^2, n=1,2 \end{aligned} \tag{13a}$$

where

$$\left|\gamma_0\right|^2 = \sigma^2 \frac{1}{1+\sqrt{1-\sigma^4}} \tag{13b}$$

Analysis of combined CE and CoE solutions shows that either both PSW modes for a structure satisfies $\left|\gamma\right|^2 < \left|\gamma_0\right|^2$ or both satisfy $\left|\gamma\right|^2 > \left|\gamma_0\right|^2$. Hence $\left|\gamma_{(-)}^{(n)}\right|^2, n=1\,\mathrm{or}\,2$ & $\left|\gamma_{(-)}^{(m)}\right|^2, m \neq n$ apply to the first structure and $\left|\gamma_{(+)}^{(n)}\right|^2, n=1\,\mathrm{or}\,2$ & $\left|\gamma_{(+)}^{(m)}\right|^2, m \neq n$ for the second. Then in the case $\kappa^{(1)} = \kappa^{(2)} < \kappa_{\max}$ the two structures are the same one with just two distinct PSW modes. For $\kappa^{(1)} = \kappa^{(2)} = \kappa_{\max}$ there is only one mode with $\left|\gamma\right|^2 = \left|\gamma_0\right|^2$.

3. MIRROR STRUCTURE SYMMETRIES FOR PSW

This section gives a brief summary of the theory for PSW to introduce the parameters and key results needed for the discussion of symmetry properties with respect to the interchange of reflection and transmission coefficients. Details of the derivations are in previous papers (McMahon, 2015b, 2016).

The scatterers are defined by a forward scattering coefficient *T* and a backward scattering coefficient *R*. For simplicity, we omit evanescent SW that may be excited by a scatterer (McMahon, 2015a). *T* and *R* are complex where $T = |T|e^{i\phi}$ and $R = |R|e^{i\chi}$ where ϕ and χ are forward and backward scattering phase shifts. *T* and *R* are satisfy $|T|^2 + |R|^2 = \sigma^2$ where $0 \le 1-\sigma^2 \le 1$.We define $|T_0|$ by $|T| = \sigma|T_0|$ and $|R_0|$ by $|R| = \sigma|R_0|$ where $|T_0|^2 + |R_0|^2 = 1$. Distinguishing PSW1 and PSW2 is their different relative signs of the phase shifts ϕ and χ for +*x* and –*x* SW travel directions. The SW wavenumber dependence of the scatterer parameters is left implicit in this paper.

The conventional approach to a theory for PSW is to derive an equation from the coupling of SW amplitudes in adjacent cells caused by forward and backward scattering. For an infinite periodic structure, the +x direction amplitude A_{n+1} and –x direction amplitude B_{n+1} for the (n+1)th cell are assumed to be equal to γA_n and γB_n respectively where γ is a single PSW amplitude independent of the cell. This leads to a quadratic equation for γ (dubbed the characteristic equation (CE)) that links γ to a complex function Γ of the scattering coefficients. For brevity the same CE for a related function $\hat{\gamma}$ can be used for both PSW types where for PSW1 $\hat{\gamma} = \gamma$ and $\hat{\gamma} = \gamma e^{-i\phi}$ for PSW2. The CE for a single PSW cannot accommodate conservation of energy for any arbitrary choice of phase shifts ϕ and χ and it was found necessary to assume different PSW amplitudes $\hat{\gamma}^{(+)}$ & $\hat{\gamma}^{(-)}$

for the +*x* and -*x* SW propagation directions (McMahon, 2016). Further, scattering couples $\hat{\gamma}^{(+)}$ & $\hat{\gamma}^{(-)}$ so they are phase correlated which can be taken into account by a phase ψ where $\hat{\gamma}^{(+)} = e^{i\psi}\hat{\gamma}^{(-)}$. This complication is overcome by a new quadratic CE found for a joint PSW pair amplitude $\tilde{\gamma}$ where $\hat{\gamma}^{(\pm)} = e^{\pm i\psi/2}\tilde{\gamma}$. Note these transformations only involve unit amplitude phase factors so that $|\tilde{\gamma}|^2 = |\hat{\gamma}^{(\pm)}|^2 = |\gamma|^2$ and expressions developed for COE previously are unaffected. It is also useful to define an energy equation (EE) for $|\tilde{\gamma}|^2$.

We find the CE, EE and their solutions, appropriate for energy flow in the +*x* direction by $|\tilde{\gamma}|^2 \le 1$, are given by[1]

$$\tilde{\gamma}^2 - 2\tilde{\Gamma}\tilde{\gamma} + 1 = 0 \tag{14a}$$

$$|\tilde{\gamma}|^4 - 2\tilde{\Delta}|\tilde{\gamma}|^2 + 1 = 0 \tag{14b}$$

$$\tilde{\Delta} = \left(\tilde{\Gamma}'^2 + \tilde{\Gamma}''^2\right) + \sqrt{\left(1 - \left(\tilde{\Gamma}'^2 + \tilde{\Gamma}''^2\right)\right)^2 + 4\tilde{\Gamma}''^2} \tag{14c}$$

$$\tilde{\gamma} = \frac{2|\tilde{\gamma}|^2}{1 + |\tilde{\gamma}|^2}\tilde{\Gamma}' - i\frac{2|\tilde{\gamma}|^2}{1 - |\tilde{\gamma}|^2}\tilde{\Gamma}'' \tag{14d}$$

$$|\tilde{\gamma}|^2 = \tilde{\Delta} - \sqrt{\tilde{\Delta}^2 - 1} \tag{14e}$$

As shown in (McMahon, 2016), Eqs.(14a-e) can be constrained by the CoE by solving two simultaneous equations of sinusoidal functions of phase $\zeta = k_s d + \phi - \psi/2$ where k_s is the SW wavenumber. The result is the equation

$$I\left(\tilde{\Delta}\right) = \left(a\bar{c} - \bar{a}c\right)^2 + \left(b\bar{c} - \bar{b}c\right)^2 - \left(b\bar{c} - \bar{b}c\right)\left(a\bar{b} - \bar{a}b\right) = 0 \tag{15}$$

where the parameters $a, b, \& c$ come from the CE and[2] $\bar{a}, \bar{b}, \& \bar{c}$ are from the CoE (see (McMahon, 2016: Eqs.(8b) and (9b))). These parameters are functions of a scattering phase difference $\varepsilon = \chi - \phi - (\pm)\pi/2$ which is zero for the BFW case. The parameters are also functions of $|\tilde{\gamma}|^2$ making Eq.(15) nonlinear generally requiring numerical solutions. Equation (15) determines all PSW solutions possible for an infinite 1D periodic structure of symmetric scatterers.

Section 4 of (McMahon, 2016) discusses how the BFW mode, incoherent wave energy model and two localized PSW modes arise from solving Eq.(15). As a typical example of numerical analysis, Fig. 1 below shows a plot of $I\left(\tilde{\Delta}\right)$ versus $\tilde{\Delta}$ from which two localized modes are seen as zeros of $I\left(\tilde{\Delta}\right)$ at $\tilde{\Delta} > 1$. There is also a zero at $\tilde{\Delta} = 1$ which is a property of pairs of phase correlated PSW. This is another extended PSW mode (besides BFW) and is discussed in more detail in Subsect.3.2.

3.1 Mirror structure symmetry properties for localized PSW modes

Figure 2 of (McMahon, 2016) plots $\sin^2(\zeta)$ and μ versus $2\varepsilon/\pi$ for localized PSW modes for three values of σ and equal reflection and transmission coefficients $|R_0| = |T_0| = 1/\sqrt{2}$. These structures are their own mirror structures and according to the CoE analysis in Subsect. 2.1 of this paper both localised modes should have the same κ. This is indeed confirmed numerically in Fig. 2 below, thereby demonstrating the consistency of Eq.(15) with the simpler analysis from the CoE alone.

---

[1] Any complex quantity q is written as $q = q' + iq''$ where q' denotes the real part and q'' denotes the imaginary part.

[2] Erratum. The parameter $|\hat{\hat{\gamma}}|^2$ in the formula for $\bar{b}$ should be $|\tilde{\gamma}|^2$.

Figure 3 demonstrates the CoE derived mirror relations for two different structures that have the same two $\kappa^{(1)} \neq \kappa^{(2)}$ for two localized PSW pairs. If we denote the two reflectivities for the structure as $\mu^{(n)}, n=1,2$ and for the mirror structure as $\bar{\mu}^{(n)}, n=1,2$ where $\bar{\mu}^{(n)}(|R_0|,|T_0|) = \mu^{(n)}(|T_0|,|R_0|), n=1,2$ then from Subsect. 2.1

$$\bar{\mu}^{(n)} = \frac{\sigma^2 - \mu^{(n)}}{1-\sigma^2\mu^{(n)}}, n=1,2 \tag{16}$$

Figure 3A plots $\kappa^{(n)}, n=1,2$ versus $2\varepsilon/\pi$ for a structure with $|R_0| = 0.4359, |T_0| = 0.9, \sigma = 1/\sqrt{2}$ where the short dashed curve is $\kappa^{(1)}$ and the solid curve is $\kappa^{(2)}$. The dotted line is $\kappa_{\max}$ and the long dashed line is $\kappa_{\min}$ and as derived in Subsect. 2.1, the $\kappa^{(n)}, n=1,2$ curves are both between these limits. Figure 3B plots $\mu^{(n)}_{(\pm)}, n=1,2$ where the red curves are from solving Eq.(15) to derive $\mu^{(n)} = \mu^{(n)}_{(+)}, n=1,2$ for the structure with $|R_0| = 0.4359, |T_0| = 0.9, \sigma = 1/\sqrt{2}$. Then applying Eq.(16) we find $\bar{\mu}^{(n)} = \mu^{(n)}_{(-)}, n=1,2$ of the mirror structure with $|R_0| = 0.9, |T_0| = 0.4359, \sigma = 1/\sqrt{2}$. The explicit solution of Eq.(15) for the mirror structure (not shown) agrees with the blue curves. Figure 3B also confirms that $\mu^{(n)}, n=1,2$ and $\bar{\mu}^{(n)}, n=1,2$ are equal distances on opposite sides of $\mu_0 = |\gamma_0|^2$ defined in Eq.(13b). Also seen from Fig. 3B that the two localized PSW modes become a single mode at $2\varepsilon/\pi$=1. This is the incoherent wave energy model result for which the reflectivity is given by (McMahon, 2015b, 2016)

$$\mu_E = \frac{\sigma^2|R_0|^2}{\frac{1}{2}(1-\sigma^4) + \sigma^4|R_0|^2 + \sqrt{\frac{1}{4}(1-\sigma^4)^2 + \sigma^4(1-\sigma^4)|R_0|^2|T_0|^2}} \tag{17}$$

It is readily verified that after interchanging the scattering coefficients $|R_0| \leftrightarrow |T_0|$ to convert Eq.(17) to a formula for $\bar{\mu}_E$, then μ_E & $\bar{\mu}_E$ satisfy the symmetry relation Eq.(16). Note also that Eq.(17) for the self-mirror case $|R_0| = |T_0| = 1/\sqrt{2}$ gives $\mu_E = \mu_0 = |\gamma_0|^2$. The maximum possible absorptivity $\kappa_E = \kappa_{\max}$ is also the self-mirror case.

3.2 Mirror structure symmetry properties for extended PSW modes

An PSW mode exposed by solutions to Eq.(15) for $\tilde{\Delta} = 1$ (see Fig. 1) requires $\tilde{\Gamma}'^2 \leq 1, \tilde{\Gamma}'' = 0$ that is only made possible by correlated pairs of PSW. However this also means $|\tilde{\gamma}|^2 = 1$ even for all $0 \leq \sigma^2 \leq 1$ but which CoE shows is physically meaningful only for $\sigma^2 = 1$. Since $|\tilde{\gamma}|^2 = 1$ this is an extended wave mode, but distinct from BFW, from which μ and η (note κ=0) can be calculated from the CE as functions of ε. The two correlated PSW are given by $\hat{\gamma}^{(\pm)} = e^{\pm i\psi/2}, \psi/2 = k_s d + \phi + \varepsilon + (\pm)\pi/2 = k_s d + \chi$, and so effectively propagate in opposite directions but generally with different SW amplitudes A and B whose ratio is derived from the CE. The reflectivity for the extended modes of correlated PSW pairs is given by

$$\mu = \frac{|R_0|^2}{1+|T_0|^2\cos(2\varepsilon) + 2\sqrt{|T_0|^2\cos^2(\varepsilon)(1-|T_0|^2\sin^2(\varepsilon))}} \tag{18}$$

The CoE equations in Subsect. 2.1 cannot derive mirror relationships for μ and $\bar{\mu}$ in this case since $\sigma^2 = 1, |\tilde{\gamma}|^2 = |\gamma|^2 = 1$, however they are found by equating expressions for $|T_0|$ or $|R_0|$ derived from Eq.(18) and the corresponding formula for $\bar{\mu}$. Figure 3 plots μ versus $2\varepsilon/\pi$ for a self-mirror structure where $\bar{\mu} = \mu$ and for two structures that are mirror related $\bar{\mu} \neq \mu$. The reflectivity for ε=0 gives $\mu = (1-|T_0|)/(1+|T_0|)$ coinciding with the centre wavenumber of the BFW passing band whereas $2\varepsilon/\pi$=1 gives $\mu = 1$ meaning the correlated pair of PSW form a standing wave.

For $\sigma^2 < 1$, Eqs.(6a,b,c) give the values μ, η and κ determined from the single solution $|\gamma|^2$ of the CE for BFW.

However the CoE analysis of Subsect. 2.1, showing that μ equates to $\left|\overleftarrow{\gamma}\right|^2$ for a mirror structure, does not apply without change for BFW because the analysis assumes the structure and its mirror have the same $\kappa = \overleftarrow{\kappa}$. Instead κ and $\overleftarrow{\kappa}$ are functions of the dimensionless phase $\xi = k_s d + \phi$ such that the interchange $|R_0| \leftrightarrow |T_0|$ produces $\overleftarrow{\kappa} \neq \kappa$ except in the self-mirror case $|R_0| = |T_0| = 1/\sqrt{2}$. Including a phase translation $\xi \rightarrow \overleftrightarrow{\xi}$ as well gives a transform that leaves κ unchanged. Hence for $\sigma^2 < 1$ the CoE analysis method of Subsect. 2.1 can then be adapted to deduce BFW results for the mirror structure from those of the original structure. The phase translation is defined by

$$\begin{aligned} \overleftrightarrow{\xi} &= \xi + \frac{\pi}{2}, 0 \le \xi \le \frac{\pi}{2} \\ \overleftrightarrow{\xi} &= \xi - \frac{\pi}{2}, \frac{\pi}{2} \le \xi \le \pi \end{aligned} \tag{19}$$

The properties of κ and $\overleftarrow{\kappa}$ under transforms $|R_0| \leftrightarrow |T_0|$ and $\xi \rightarrow \overleftrightarrow{\xi}$ can be derived from the CE and are

$$\kappa(\xi) \equiv \kappa(|R_0|, |T_0|, \xi) = \kappa(|T_0|, |R_0|, \overleftrightarrow{\xi}) \equiv \overleftarrow{\kappa}(\overleftrightarrow{\xi}) \tag{20a}$$

$$\overleftarrow{\kappa}(\xi) \equiv \kappa(|T_0|, |R_0|, \xi) = \kappa(|R_0|, |T_0|, \overleftrightarrow{\xi}) \equiv \kappa(\overleftrightarrow{\xi}) \tag{20b}$$

Equations (20a,b) express an invariance property for BFW. Figure 5 demonstrates this property where the grey curve in Fig. 5B is $\kappa(|T_0|, |R_0|, \overleftrightarrow{\xi})$ that coincides exactly with $\kappa(|R_0|, |T_0|, \xi)$ which is the blue curve of Fig. 5A. Also demonstrated is the constraint Eq.(8) $\kappa_{\min} \le \kappa(|R_0|, |T_0|, \xi) \le \kappa_{\max}$ satisfied by the curves in Fig. 5A. The maximum absorptivity $\kappa(|R_0|, |T_0|, \xi) = \kappa_{\max}$ is reached for at most two dimensionless wavenumbers in the range 0 to π which can be shown to only occur for $\sqrt{(1-\sigma^2)/2} \le |T_0| \le \sqrt{(1+\sigma^2)/2}$.

Now that Eq.(20,a,b) define how two functions $\kappa(|R_0|, |T_0|, \xi)$ and $\kappa(|R_0|, |T_0|, \overleftrightarrow{\xi})$ connect two mutually mirror structures, two pairs of functions $\left|\gamma_{(\pm)}\right|^2$ (only one of which is a true BFW $|\gamma|^2$) and $\mu_{(\mp)} = \left|\gamma_{(\pm)}\right|^2$ (the one that is not $|\gamma|^2$ is the true BFW μ) can be defined similar to Eqs. (7) and (11) and relate the properties of these two structures. Generalising Eq.(11) the BFW reflectivity of the mirror structure $\overleftarrow{\mu}(\xi) \equiv \mu(|T_0|, |R_0|, \xi)$ is related to the structure reflectivity $\mu(\xi) \equiv \mu(|R_0|, |T_0|, \xi)$ by

$$\overleftarrow{\mu}(\xi) = \frac{\sigma^2 - \mu(\overleftrightarrow{\xi})}{1 - \sigma^2 \mu(\overleftrightarrow{\xi})} \tag{21}$$

Equation (21) is demonstrated by Fig. 6. In 6A the solid blue curve plots $\mu(\xi) \equiv \mu_{(-)}(\xi)$ and the dashed blue curve plots $(\sigma^2 - \mu(\xi))/(1 - \sigma^2 \mu(\xi)) \equiv \mu_{(+)}(\xi)$ which is equivalent to $|\gamma(\xi)|^2$. For 6B, the solid black curve plots $\overleftarrow{\mu}(\xi) \equiv \overleftarrow{\mu}_{(-)}(\xi)$ and the dashed black curve plots $(\sigma^2 - \overleftarrow{\mu}(\xi))/(1 - \sigma^2 \overleftarrow{\mu}(\xi)) \equiv \overleftarrow{\mu}_{(+)}(\xi)$. Then 6C shows the two phase translated curves derived from 6B reproduces the two curves of 6A. Hence the solid black curve $\overleftarrow{\mu}(\overleftrightarrow{\xi})$ of 6C equates to blue dashed curve of 6A. This is just the phase translated version of Eq.(21) and hence Fig. 6 demonstrates Eq.(21).

Equation (21) relies on at least slightly inelastic scattering and applies for $1 - \sigma \rightarrow 0$ but not $1 - \sigma = 0$ where the RHS of Eq.(21) becomes 1 which is only correct inside a BFW stopping band, and cannot account for the ξ dependence of $\overleftarrow{\mu}(\xi)$ in the passing band. This implies a discontinuity in PSW phenomena where scattering becomes slightly inelastic and requires further analysis.

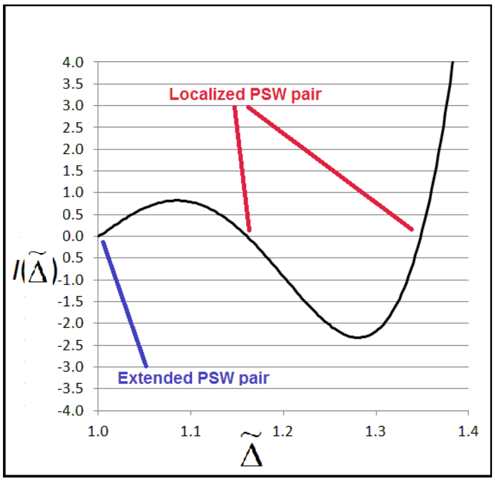

Figure 1: Plot of the function $I(\tilde{\Delta})$ for the self-mirror structure $|T_0| = 1/\sqrt{2}, \sigma = 0.90$, at the phase shift $2\varepsilon/\pi = 0.9$

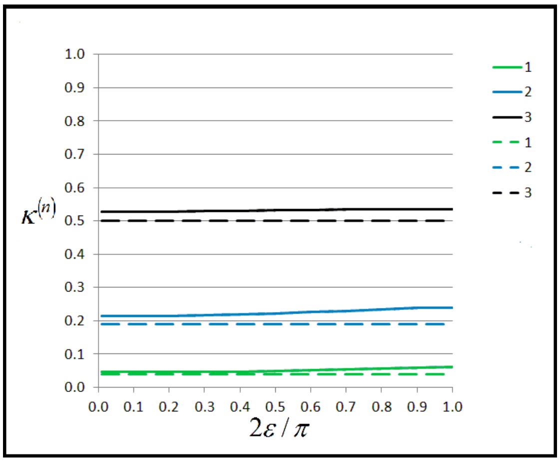

Figure 2: Plots of absorptivity $\kappa^{(n)}, n = 1,2$ versus $2\varepsilon/\pi$ for the two localized PSW modes of self-mirror periodic structures. The solid lines are for one mode, dashed lines for the other mode are not visible since $\kappa^{(1)} = \kappa^{(2)}$, and the long dashed lines show $\kappa_{\min}$. 1. $|T_0| = 1/\sqrt{2}, \sigma = 0.98$ 2. $|T_0| = 1/\sqrt{2}, \sigma = 0.90$ 3. $|T_0| = 1/\sqrt{2}, \sigma = 1/\sqrt{2}$.

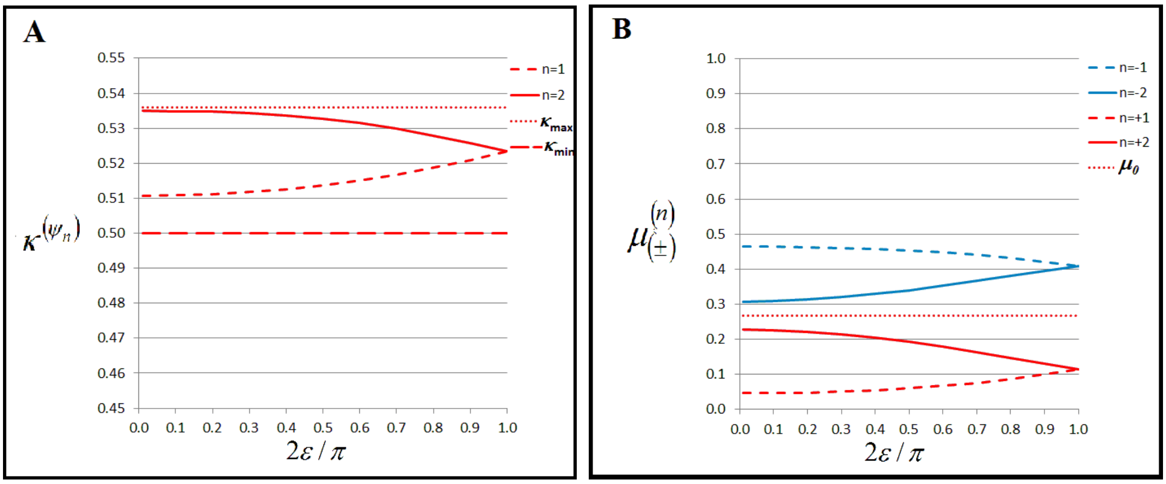

Figure 3: Four correlated pairs of PSW for unequal reflection and transmission coefficients of two mirror related structures, one with $|R_0| = 0.4359, |T_0| = 0.9, \sigma = 1/\sqrt{2}$ and the other with $|R_0| = 0.9, |T_0| = 0.4359, \sigma = 1/\sqrt{2}$.

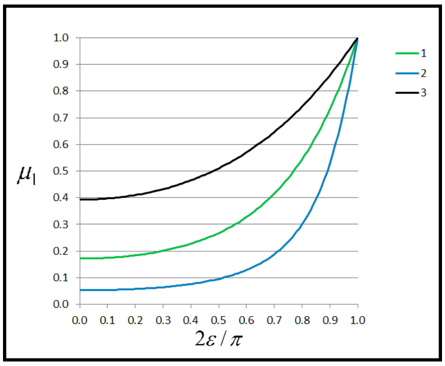

Figure 4: Extended wave reflectivities by correlated pairs of PSW that are predicted to be possible for elastic scattering $\sigma = 1$. 1. $|T_0| = |R_0| = 1/\sqrt{2}$, 2. $|T_0| = 0.9, |R_0| = 0.436$, 3. $|T_0| = 0.436, |R_0| = 0.9$.

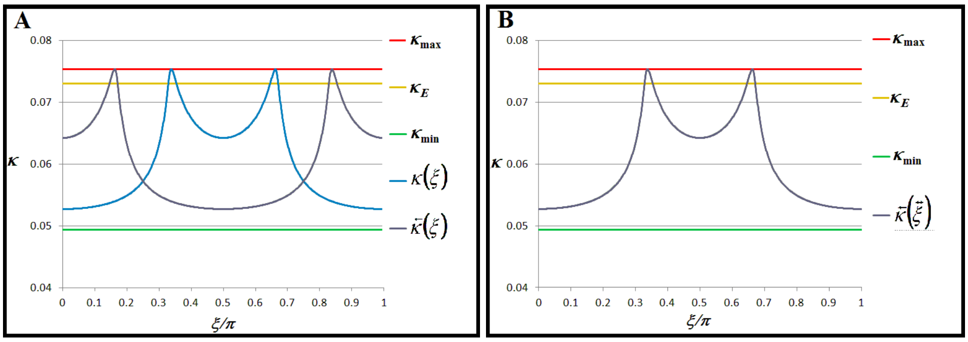

Figure 5: Plots of $\kappa(\xi)$, $\bar{\kappa}(\xi)$ & $\bar{\kappa}(\vec{\xi})$ for $\sigma = 0.975$ demonstrating the mirror symmetry of BFW. A. Blue curve $\kappa(0.8660, 0.5, \xi)$. Grey curve $\kappa(0.5, 0.8660, \xi)$ B. $\kappa(0.5, 0.8660, \vec{\xi})$.

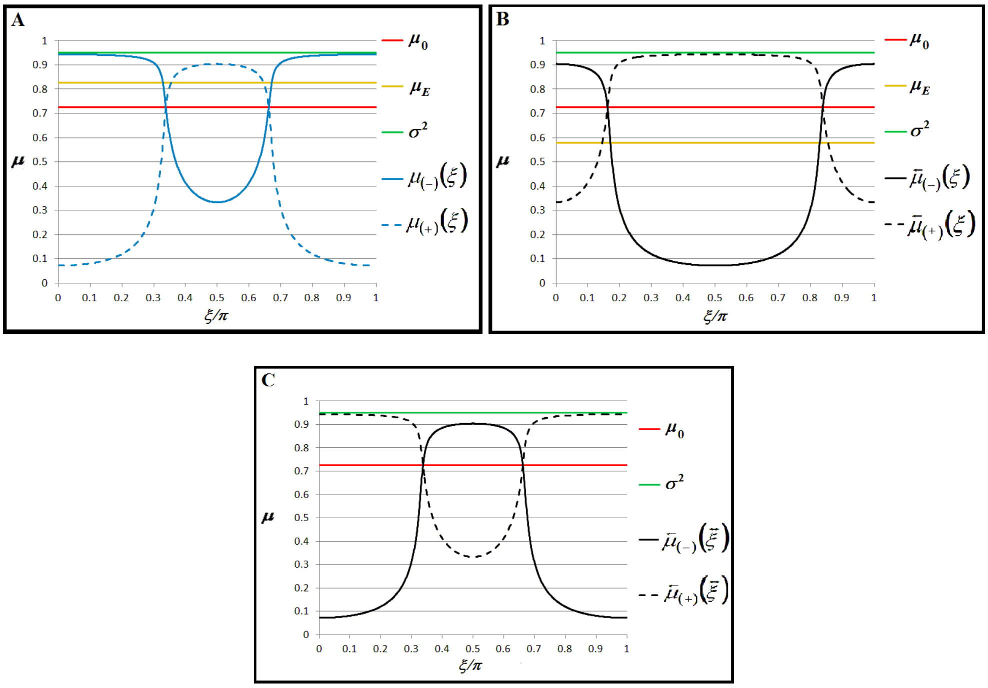

Figure 6: Plots of $\mu_{(\pm)}(\xi)$, $\bar{\mu}_{(\pm)}(\xi)$ & $\bar{\mu}_{(\pm)}(\vec{\xi})$ for $\sigma = 0.975$ demonstrating the mirror symmetry of BFW. A. $|R_0| = 0.8660, |T_0| = 0.5$, B. $|R_0| = 0.5, |T_0| = 0.8660$, C. phase translated curves of B.

4. DISCUSSION

For such a simple system of plane SW travelling back and forth between an infinite number of equally spaced scatterers, a variety of periodic structure wave phenomena is found. The well-known Bloch-Floquet waves arise where the difference of backward and forward scattering phase shifts is $\pm\pi/2$. Any other phase shifts eliminate the possibility of BFW although as shown in Subsect. 3.2 another type of extended wave can exist for pairs of correlated PSW mutually coupled by elastic scattering. Whereas BFW are inherently consistent with CoE and inelastic scattering where $\sigma^2 < 1$, other phase shifts require explicit fusion of CoE with the CE which is not possible for a single PSW like BFW but requires pairs of correlated PSW. A consequence of inelastic collisions and non-BFW scattering phase shifts is that only localized PSW modes are possible. A maximum of two localized PSW modes is predicted by the theory embodied in Eq.(15).

Although the fusion of CoE with the CE leads to a somewhat complicated Eq.(15), some of the results can be understood in simpler terms from CoE as shown in Subsect. 2.1. Two PSW modes can have the same periodic structure absorptivity κ resulting in simple relationships between the properties of a periodic structure and its mirror structure defined by interchanging the scatterer reflection and transmission coefficients. The symmetry of structure and its mirror structure absorptivity and reflectivity is demonstrated in Subsects. 3.1 and 3.2 for localized PSW and BFW respectively.

Interestingly, the incoherent wave energy model gives the same results as a localized mode from pairs of correlated PSW where the difference of scattering phase shifts equals $\pm\pi$. Perhaps randomisation of the phases of SW at any point owing to contributions from scattering from many scatterers around that point is a valid alternative interpretation of the localized modes, however simulations or explicit summation of the contributions from multiple scattering would be needed to confirm this idea.

Finally it should be said that unlike BFW the extra extended and localized PSW modes of this paper are predictions and so far lack confirmation by experiments. The key requirement is scatterers with phase shift differences that deviate from those for BFW, namely $\chi - \phi \neq \pm\pi/2$. Such phase shifts can arise for scatterers that are internally asymmetric however the reflection, transmission and energy absorption coefficients would then be different for opposite directions of SW propagation requiring for self-consistency an extension of the theory to asymmetric periodic structures. Such investigations might lead to practical developments, such as passive damping of energy propagation along periodic structures by designing them to only support localized modes.

ACKNOWLEDGEMENT

This research was partially supported by the Australian Defence Science and Technology Group.

REFERENCES

Brillouin, L 1953, *Wave Propagation in Periodic Structures*, 2nd edn, Dover, Inc.

McMahon, D 2015a, 'Bloch-Floquet flexural waves for an infinite ribbed plate including near field effects', *Proceedings of the 22nd International Congress on Sound and Vibration* (ICSV22), Florence, Italy.

McMahon, D 2015b, 'Wave propagation in infinite periodic structures taking into account energy absorption', *Proceedings of Acoustics 2015*, Hunter Valley, Australia.

McMahon, D 2016, 'Wave propagation in infinite periodic structures', *Proceedings of the 23rd International Congress on Sound and Vibration* (ICSV23), Athens, Greece.